COMPARING COMBINATORIAL MODELS OF MODULI SPACE AND THEIR COMPACTIFICATIONS

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Abstract. In this paper we compare two combinatorial models for the moduli space of two-dimensional cobordisms: Bödigheimer's radial slit configurations and Godin's admissible fat graphs, producing an explicit homotopy equivalence using a "critical graph" map. We also discuss natural compactifications of these two models, the unilevel harmonic compactification and Sullivan diagrams respectively, and prove that the homotopy equivalence induces a cellular homeomorphism between these compactifications.

Contents

1. Introduction

In this paper we compare two combinatorial models of the moduli space of cobordisms, so we start this section with an introduction to moduli space, giving a conformal description of it. After that we describe various combinatorial models and how they relate to each other, which includes our main result, Theorem 1.1. Finally we describe two applications.

1.1. **The moduli space of cobordisms.** Mathematicians have been interested in surfaces and their properties for centuries. An integral part of this, the study of families of surfaces, known as "moduli theory", goes back to the nineteenth century. This study can proceed along many different paths – one can use algebraic geometry, hyperbolic geometry, complex geometry, conformal geometry or group theory – and the resulting interplay led to large amounts of interesting mathematics. One of the main points of this theory is the construction of *moduli spaces*. Intuitively a moduli space is a space of all surfaces isomorphic to a given one, characterized by the property that equivalence classes of maps into it correspond to equivalence classes of families of surfaces.

For modern applications to field theories, the type of surfaces of interest is that of two-dimensional oriented cobordisms. A two-dimensional oriented cobordism is an oriented surface S with parametrized boundary divided into an incoming and outgoing part. More precisely, there is a pair of maps

$$\iota_{\text{in}} : \bigsqcup_{i=1}^{n} S^1 \to \partial S \qquad \text{and} \qquad \iota_{\text{out}} : \bigsqcup_{j=1}^{m} S^1 \to \partial S$$

such that $\iota_{\text{in}} \sqcup \iota_{\text{out}}$ is a diffeomorphism onto ∂S.

Studying operations indexed by isomorphism classes of such two-dimensional oriented cobordisms leads to the definition of two-dimensional TQFT's. If one is interested in more refined structure, one can take into account the entire moduli space of such cobordisms, instead of just its connected

Date: September 28, 2015.

Alexander Kupers is supported by a William R. Hewlett Stanford Graduate Fellowship, Department of Mathematics, Stanford University.

components. We will now give the analogue of the conformal definition of the moduli space of these cobordisms, following section 2 of [Böd06] and [Ham13].

Let S be an isomorphism class of connected two-dimensional oriented cobordism with non-empty incoming and outgoing boundary. As we will later endow S with a metric, we can think of the parametrization of its boundary as being given by a point in each boundary component. So $S = S_{g,n+m}$ is a connected oriented surface of genus g with $n+m$ boundary components, each containing a single point p_i for $1 \leq i \leq n+m$. The marked points are ordered and divided into an incoming set (which contains $n \geq 1$ marked points) and an outgoing set (which contains $m \geq 1$ marked points).

To define moduli space we start by considering the set of metrics g on S. Two metrics are said to be conformally equivalent if they are equal up to a pointwise rescaling by a continuous function. This is equivalent to having the same notion of angle.

A diffeomorphism $f : S_1 \to S_2$ between two-dimensional manifolds $(S_1, [g]_1)$, $(S_2, [g]_2)$ with conformal classes of metrics is said to be a conformal diffeomorphism if $f^*[g]_2 = [g]_1$. This is equivalent to its differential $D_p f$ being a linear map that preserves angles at each $p \in S_1$.

We want to restrict our attention to those conformal classes of metrics on S that have the following property: each incoming boundary component has a neighborhood that is conformally diffeomorphic to a neighborhood of the boundary of $\{z \in \mathbb{C} \,|\, ||z|| \geq 1\}$ and each outgoing boundary component has a neighborhood that is conformally diffeomorphic to a neighborhood of the boundary of $\{z \in \mathbb{C} \,|\, ||z|| \leq 1\}$. We say that these conformal classes have good boundary.

The moduli space $\mathcal{M}_g(n,m)$ will have as underlying set the set of conformal classes of metrics on S with good boundary modulo the relation of conformal diffeomorphism fixing the points p_i. We will now define the Teichmüller metric on this set. In this metric two equivalence classes of metrics on S are close together if they are related by a homeomorphism that outside a finite set is not only differentiable, but in fact conformal up to a small error. To make this precise, note that a linear map $D : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal – that is, preserves angles – if and only if $\max \frac{||Dv||}{||v||} = \min \frac{||Dv||}{||v||}$. Hence we can quantify the deviation of a linear map from being conformal by its eccentricity:

$$\mathrm{Ecc}(D) := \frac{\max ||Dv||/||v||}{\min ||Dv||/||v||}$$

Let $f : (S, [g]_1) \to (S, [g]_2)$ be a homeomorphism that is continuously differentiable outside a finite set of points Σ, we call this a quasi-conformal homeomorphism. Its quasi-conformal constant K_f is defined to be

$$K_f := \sup_{p \in S \setminus \Sigma} \mathrm{Ecc}(D_p f)$$

and f is said to be quasi-conformal if K_f is finite. If $QC([g]_1, [g]_2)$ denotes the set of all quasiconformal homeomorphisms between $(S, [g]_1)$ and $(S, [g]_2)$ fixing the points p_i, then we can define the Teichmüller distance between $[g]_1$ and $[g]_2$ as follows:

$$d_{\mathcal{T}}((S, [g]_1), (S, [g]_2)) = \log \inf\{K_f \,|\, f \in QC([g]_1, [g]_2)\}$$

This completes the conformal definition of the moduli space of two-dimensional oriented cobordisms isomorphic to S. It is the metric space defined as follows:

$$\mathcal{M}_g(n,m) = \left(\frac{\text{conformal classes of metrics on } S \text{ with good boundary}}{\text{conformal diffeomorphisms fixing the points } p_i}, \text{ Teichmüller metric} \right)$$

For S that are not connected, we simply take the product of these spaces over all components. An alternative definition of these spaces is as the quotient of Teichmüller space (the space of quasiconformal maps modulo conformal equivalence) by the action of the mapping class group $\mathrm{Mod}(S, \partial S)$ (the connected components of the diffeomorphism group $\mathrm{Diff}(S, \partial S)$). This is a free proper action on a contractible space and hence $\mathcal{M}_g(n,m) \simeq B\mathrm{Mod}(S, \partial S)$. In that case all connected components of $\mathrm{Diff}(S, \partial S)$ are contractible and we can thus furthermore conclude that

$$\mathcal{M}_g(n,m) \simeq B\mathrm{Mod}(S, \partial S) \simeq B\mathrm{Diff}(S, \partial S)$$

The last term makes clear why $\mathcal{M}_g(n,m)$ is a model for the moduli space of two-dimensional oriented cobordisms; any bundle of cobordisms over a paracompact space B with transition functions given by diffeomorphisms, can be obtained by pulling back a universal bundle from $\mathcal{M}_g(n,m)$ along a map $B \to \mathcal{M}_g(n,m)$.

We end this subsection by describing this universal bundle. Just like the moduli space $\mathcal{M}_g(n,m)$ is the quotient of the space of conformal classes of metrics $[g]_1$ by the diffeomorphisms, the universal bundle is the quotient of the space consisting of pairs $([g],x)$ of a conformal class of metrics and a point $x \in S$, by conformal diffeomorphisms acting diagonally.

1.2. **An overview of combinatorial models of moduli space.** The goal of this paper is to discuss several combinatorial models of the moduli space of cobordisms with non-empty incoming and outgoing boundary and their compactifications.

The following diagram spells out the relations between various models discussed in this paper. An arrow is a continuous map; if decorated by $\simeq$ it is homotopy equivalence, if it is double-headed it is a surjection, and if decorated by $\cong$ it is a homeomorphism. We fix g, n and m and drop them from the notation:

$$\begin{tikzcd}
\mathcal{M} & & & \\
\mathbf{RAD} \arrow[u, "\cong"] \arrow[d, "\simeq"'] & & {|\mathcal{F}at^{ad}|} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & {|\mathcal{F}at|} \arrow[d, "\simeq"] \\
\mathfrak{Rad} \arrow[d, "\text{compactification}"] & \mathfrak{Rad}^{\sim} \arrow[l, two heads, "\simeq"', "(4.1)"] \arrow[r, "\simeq", "(4.3)"'] & \mathcal{MF}at^{ad} \arrow[r, "\simeq"] \arrow[dd, "\text{quotient by slides}"] & \mathcal{MF}at \\
\overline{\mathfrak{Rad}} \arrow[d, "\simeq"'] & & & \\
\overline{\mathfrak{URad}} \arrow[rr, "(5)", "\cong"'] & & \mathcal{SD} &
\end{tikzcd}$$

Let us summarize the objects that are appear in this diagram.

Moduli space $\mathcal{M}$: This is the archetypical "space of cobordisms," a conformal model of which was discussed in Section 1.1. It consists of conformal classes of metrics modulo conformal diffeomorphisms, with the Teichmüller metric.

The radial slit configurations RAD and $\mathfrak{Rad}$: This is a model for $\mathcal{M}$ due to Bödigheimer, consisting of glueing data to construct a conformal class of metric by glueing together annuli in $\mathbb{C}$. The main theorem of [Böd06] is that there exist maps $\mathcal{M} \to \mathbf{RAD}$ and $\mathbf{RAD} \to \mathcal{M}$ which are mutually inverse homeomorphisms. There is a deformation retraction of **RAD** onto $\mathfrak{Rad}$ by fixing the radii of the annuli. This and related models will be discussed in detail in Section 2, and $\mathfrak{Rad}$ will be defined in Definition 2.16.

The fat graphs $\mathcal{F}at$: Fat graphs are graphs with the additional structure of a cyclic ordering of the edges going into each vertex and data encoding the parametrization of its "boundary components." Taking as morphisms maps of fat graphs that collapse a disjoint union of trees defines a category of fat graphs $\mathcal{F}at$. The space $|\mathcal{F}at|$ is the geometric realization of this category. This and related models will be discussed in detail in Section 3, and $\mathcal{F}at$ will be defined in Definition 3.7.

The admissible fat graphs $\mathcal{F}at^{ad}$: A fat graph is said to be admissible if its incoming boundary graph embeds in it. The space $|\mathcal{F}at^{ad}|$ is the geometric realization of the full subcategory on the admissible fat graphs. It is defined in Definition 3.7.

The metric fat graphs $\mathcal{MF}at$: Closely related to $\mathcal{F}at$ is the space of metric fat graphs $\mathcal{MF}at$. This is the space of fat graphs with the additional data of lengths of their edges. The topology is described in terms of these lengths and it contains the realization of $\mathcal{F}at$ as a deformation retract.

The admissible metric fat graphs $\mathcal{MF}at^{ad}$: Just like $\mathcal{F}at^{ad}$ is the subcategory of $\mathcal{F}at$ consisting of fat graphs that are admissible, $\mathcal{MF}at^{ad}$ is the subspace of $\mathcal{MF}at$ consisting of metric fat graphs that are admissible. It is defined in Definition 3.11.

- **The fattening of the radial slit configurations $\mathfrak{Rad}^\sim$:** To discuss the relation between $\mathfrak{Rad}$ and $\mathcal{MF}at$, in this paper we introduce $\mathfrak{Rad}^\sim$ as a thicker version of $\mathfrak{Rad}$ by including resolutions of the critical graph for non-generic radial slit configurations. This is done in Subsection 4.1.
- **The harmonic compactification $\overline{\mathfrak{Rad}}$:** Naturally $\mathfrak{Rad}$ arises as an open subspace of a compact space $\overline{\mathfrak{Rad}}$. In this compactification we allow identifications of points on the outgoing boundary and allow handles to degenerate to intervals. It is defined in Definition 2.16.
- **The unilevel harmonic compactification $\overline{\mathfrak{URad}}$:** The space $\overline{\mathfrak{URad}}$ is a deformation retract of $\overline{\mathfrak{Rad}}$ obtained by making all slits equal length. It is defined in Definition 2.22.
- **The Sullivan diagrams $\mathcal{SD}$:** The space of Sullivan diagrams are the quotient of $\mathcal{MF}at^{ad}$ by the equivalence relation of slides away from the admissible boundary. It is defined in Definition 3.16.

In this article we will focus on the bottom square; that is, the relations between radial slit configurations, admissible metric fat graphs and their compactifications. Our main result is the following:

Theorem 1.1. *We define a space $\mathfrak{Rad}^\sim$ and maps* (4.19)*,* (4.28) *and* (5.1) *such that there is a commutative square*

$$\begin{tikzcd}
\mathfrak{Rad} \arrow[d] & \mathfrak{Rad}^\sim \arrow[l, "\simeq", "(4.19)"'] \arrow[r, "\simeq", "(4.28)"'] & \mathcal{MF}at^{ad} \arrow[dd] \\
\overline{\mathfrak{Rad}} \arrow[d, "\simeq"', "(2.23)"] & & \\
\overline{\mathfrak{URad}} \arrow[rr, "(5.1)", "\cong"'] & & \mathcal{SD}
\end{tikzcd}$$

Furthermore, all maps that are decorated by $\simeq$ are homotopy equivalences and the map decorated by $\cong$ is a cellular homeomorphism.

There exist other combinatorial models related to the moduli space of cobordisms which are not discussed in detail in this paper. We will describe six such models in the following remarks.

Remark 1.2. In order to describe an action of the chains of the moduli space of surfaces on the Hochschild homology of $\mathscr{A}_\infty$-Frobenius algebras, Costello constructs a chain complex that models the homology of the moduli space ([Cos07a, Cos07b]). In [WW11], Wahl and Westerland describe this chain complex in terms of fat graphs with two types of vertices, which they denote *black and white fat graphs*. There is an equivalence relation of black and white graphs given by slides away from the white vertices. The quotient chain complex is the cellular chain complex of $\mathcal{SD}$. Furthermore, in [ES14] it is shown that $\mathcal{MF}at^{ad}$ has a quasi-cell structure of which *black and white fat graphs* is its cellular complex and the quotient map to $\mathcal{SD}$ respects this cell structure.

Remark 1.3. In [CG04] Cohen and Godin define *Sullivan chord diagrams* of genus g with p incoming and q outgoing boundary components. These chord diagrams were also used in [FT09]. These are fat graphs obtained from glueing trees to circles. These fit together into a space $\mathcal{CF}(g;p,q)$ and this space is a subspace of $\mathcal{MF}at^{ad}$. They are thus *not* the same as Sullivan diagrams, here defined in Definition 3.16, though they do admit a map to $\mathcal{SD}$. It is known that the space of metric chord diagrams is not homotopy equivalent to moduli space, see remark 3 of [God07a].

Remark 1.4. In her thesis [Poi10], Poirier defines a space $\overline{SD}(g,k,l)/\!\sim$ of *string diagrams modulo slide equivalence* of genus g with k incoming and l outgoing boundary components and more generally she defines *string diagrams with many levels modulo slide equivalence* $\overline{LD}(g,k,l)/\!\sim$. Proposition 2.3 of [Poi10] says that $\overline{SD}(g,k,l)/\!\sim \;\simeq \overline{LD}(g,k,l)/\!\sim$. She also defines a subspace $SD(g,k,l)$ of $\overline{SD}(g,k,l)$. Both $\overline{SD}(g,k,l)$ and $SD(g,k,l)$ are subspaces of $\mathcal{MF}at^{ad}$ and by counting components one can see that these inclusions can not be homotopy equivalences. However, there is an induced map $\overline{SD}(g,k,l)/\!\sim \;\to \mathcal{SD}$ which is a homeomorphism.

Remark 1.5. In [DCPR15] Drummond-Cole, Poirier, and Rounds define a space of *string diagrams* SD which generalize the spaces of chord diagrams constructed in [Poi10] They conjecture that this space is homotopy equivalent to the moduli space of Riemann surfaces. In this direction, one can embed $SD \hookrightarrow \mathcal{MFat}^{ad}$. However, it is not clear from the results of this paper that this embedding is a homotopy equivalence. Furthermore, there is an equivalence relation $\sim$ on SD, which is not discussed in their paper, and they conjecture that $SD/\sim$ is homotopy equivalent to the harmonic compactification of moduli space.

Remark 1.6. Following the ideas of Wahl, Klamt constructs a chain complex of *looped diagrams* denoted $l\mathcal{D}$ in [Kla13]. This complex gives operations on the Hochschild homology of commutative Frobenius algebras. These operations are natural in the algebra and assemble to something like a field theory. Moreover, she gives a chain map from cellular complex of the space of Sullivan diagrams to looped diagrams. However, a geometric interpretation of a space underlying the complex $l\mathcal{D}$ and its possible relation to moduli space is still unknown.

Remark 1.7. In [Kau10], Kaufman describes a space of open-closed Sullivan diagrams $\mathrm{Sull}_1^{c/o}$ in terms of arcs embedded in a surface. The closed part, Sull_1^{c}, is a space whose points correspond to weighted families of embedded arcs in the surface that flow from the incoming boundary to the outgoing boundary. This space has a natural cell structure and one can construct a cellular homeomorphism $\mathrm{Sull}_1^{c} \xrightarrow{\cong} \mathcal{SD}$ in a way that is made precise in [WW11, Remark 2.12] .

1.3. **Applications of these models.** We will next explain two of the applications of combinatorial models for moduli space.

1.3.1. *Explicit computations of the homology of moduli spaces.* We will see that combinatorial models provide cell decompositions for moduli space. This make an explicit computation of the (co)homology of moduli space using cellular (co)homology possible. Instead of studying $\mathcal{M}_g(n,m)$, it turns out to be more convenient to study the closely related moduli space $\mathcal{M}_g^{1,n}$ of surfaces of genus g with one parametrized boundary component and permutable n punctures. There are variations of $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$ that are models for $\mathcal{M}_g^{1,n}$.

Simultaneously, much is known about the homology of $\mathcal{M}_g^{1,n}$ and much is unknown about it. In particular, Harer stability tells us $H_*(\mathcal{M}_g^{1,n})$ stabilizes as $g \to \infty$ or $n \to \infty$ [Har85, Wah13] and the Madsen-Weiss theorem allows you to compute the stable homology with field coefficients [MW05, Gal04]. On the other hand, we know almost nothing of the homology outside of the stable range, except that there has to be a lot of it. Explicit computations of the homology of $\mathcal{M}_g^{1,n}$ for low g and n are helpful to inform and test conjectures about the general structure of the homology of moduli spaces.

The computation of the homology of moduli spaces using radial slit configurations, or the closely related parallel slit configurations, is a long-term project of Bödigheimer and his students. The first example of this is Ehrenfried's thesis [Ehr98] where he computes $\mathcal{M}_2^{1,0}$. See [ABE08] for a report of the computations of the integral homology of $\mathcal{M}_g^{1,n}$ for $2g+n \leq 5$ using parallel slits.

An example of an explicit computation of the homology of moduli spaces using fat graphs is given in [God07b]. There Godin computes the integral homology of the $\mathcal{M}_g^{1,0}$ for $g = 1,2$ and $\mathcal{M}_g^{2,0}$ for $g = 1$.

1.3.2. *Two-dimensional field theories, in particular string topology.* Combinatorial models of moduli space have been an important tool in the study of two-dimensional field theories for a long time. One of the first applications was Kontsevich's proof of the Witten conjecture using fat graphs [Kon92]. Since then people have used fat graphs to understand field theories, one example of which is Costello's classification of so-called classical conformal field theories [Cos07b].

More concretely combinatorial models for the moduli space of cobordisms have played a big role in the construction of string operations; these are operations $H_*(\mathcal{M}_g(n,m);\mathcal{L}^{\otimes d}) \otimes H_*(LM)^{\otimes n} \to H_*(LM)^{\otimes m}$ for compact oriented manifolds M. Chas and Sullivan thought of the pair of pants cobordism as a figure-eight graph [CS99], and many of the constructions of string operations since have used graphs. An important example is Godin's work [God07a], which uses $\mathcal{Fat}^{ad}$. Using Costello's model for moduli space together with a Hochschild homology model for the cohomology of the free loop space, Wahl and Westerland [WW11, Wah12] not only constructed string operations,

but showed that these factor through $\mathcal{SD}$. One can also use radial slit configurations to construct string operations.

One problem in string topology is that there is a large amount of different constructions, but few comparisons between these constructions. The critical graph equivalence of Section 4 might make it possible to compare constructions involving fat graphs and Sullivan diagrams, to constructions involving radial slit configurations and the harmonic compactification.

1.4. **Outline of paper.** In Sections 2 and 3 we define radial slit configurations, (metric) fat graphs and their compactifications in detail. In Section 4 we prove that the critical graph of a radial slit configuration allows one to construct a zigzag of homotopy equivalences between $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$. In Section 5 we prove that this homotopy equivalence descends to a homeomorphism between $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$.

1.5. **Acknowledgments.** This paper grew out of discussions at the String Topology and Related Topics at the Center for Symmetry and Deformation at the University of Copenhagen and was finished during the Hausdorff Trimester Program on Homotopy Theory, Manifolds, and Field Theories. The authors would like to thank Carl-Friedrich Bödigheimer and Nathalie Wahl for helpful conversations and comments. The first author was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92). The second author was supported by a William R. Hewlett Stanford Graduate Fellowship, Department of Mathematics, Stanford University.

2. Radial slit configurations and the harmonic compactification

2.1. **The definition.** In this subsection we introduce Bödigheimer's radial slit configuration model for the moduli space of two-dimensional cobordisms with non-empty incoming and outgoing boundary. The idea is that any such cobordism and any conformal class of metric on it can be obtained by taking annuli, making cuts in these annuli and then glueing along these cuts.

All of the material in this subsection is due to Bödigheimer, and references for the radial slit configurations and related models include [Böd90], [Böd06], [ABE08], [Ebe03] and [Böd07]. The last one may be of particular interest to the reader, as it describes in a closely related setting a more elegant alternative construction to the one outlined below, using subspaces of bar complexes associated to symmetric groups. It however leads to a different compactification of moduli space than the harmonic compactification, so we prefer to use [Böd06].

2.1.1. *Spaces of radial slit configurations.* Before giving a rigorous definition of the radial slit configuration space $\mathfrak{Rad}$ we explain how to arrive at this definition given that one wants to build cobordisms from glueing annuli along cuts. The reader may skip this introduction and go directly to Definition 2.1, the definition of the possibly degenerate radial slit preconfigurations.

The simplest cobordism with non-empty incoming and outgoing boundary is the cylinder, with one incoming and one outgoing boundary component. Using the theory of harmonic functions, one can see that each annulus is conformally equivalent to one of the following annuli for $R \in (\frac{1}{2\pi}, \infty)$ [Ham13, Corollary 2.13] (the reason for the choice of $\frac{1}{2\pi}$ is to facilitate comparison with fat graphs later on):

$$\mathbb{A}_R = \left\{ z \in \mathbb{C} \,\middle|\, \frac{1}{2\pi} \leq |z| \leq R \right\}$$

We will therefore take these as our basic building blocks. Each of them has an inner boundary $\partial_{\text{in}}\mathbb{A}_R = \{z \in \mathbb{C} \,|\, |z| = \frac{1}{2\pi}\}$ and outer boundary $\partial_{\text{out}}\mathbb{A}_R = \{z \in \mathbb{C} \,|\, |z| = R\}$. They also come with a canonical metric, being subsets of the complex plane.

Suppose we are interested in creating a cobordism with n incoming boundary components. Then we will start with an ordered disjoint union of n annuli $\mathbb{A}^{(i)}_{R_i}$, whose inner boundaries are going to be the incoming boundary of our cobordism. To construct our cobordism we will make cuts radially inward from the outer boundaries of the annuli. Such cuts are uniquely specified by points $\zeta \in \sqcup_{i=1}^n \mathbb{A}^{(i)}_{R_i}$, which we will call slits. They need not be disjoint. As will become clear, the number of slits must always be an even number $2h$ and we thus number them $\zeta_1, \ldots, \zeta_{2h}$. It turns out that for a total genus g cobordism with n incoming and m outgoing boundary components we will need $2h = 2(2g - 2 + n + m)$ slits.

We want to glue the different sides of the cuts together to get back a surface. To get a metric on the surface from the metric on the cut annuli, necessarily two cuts that we glue together must be of the same length. For the orientations to work out, we must glue a side clockwise from a cut to a side counterclockwise from a cut. To avoid singularities, if one side of the cut corresponding to ζ_i is glued to a side of the cut corresponding to ζ_j, the same must be true for the other two sides. From this we see that our glueing procedure should be described by a pairing on $\{1, \ldots, 2h\}$, encoded by a permutation $\lambda : \{1, \ldots, 2h\} \to \{1, \ldots, 2h\}$ consisting of h cycles of length 2. We should furthermore demand that if ζ_i lies of the annulus $\mathbb{A}^{(j)}_{R_j}$ and $\zeta_{\lambda(i)}$ lies on the annulus $\mathbb{A}^{(j')}_{R_{j'}}$, then $R_j - |\zeta_i| = R_{j'} - |\zeta_{\lambda(i)}|$. See Figure 1 for a simple example.

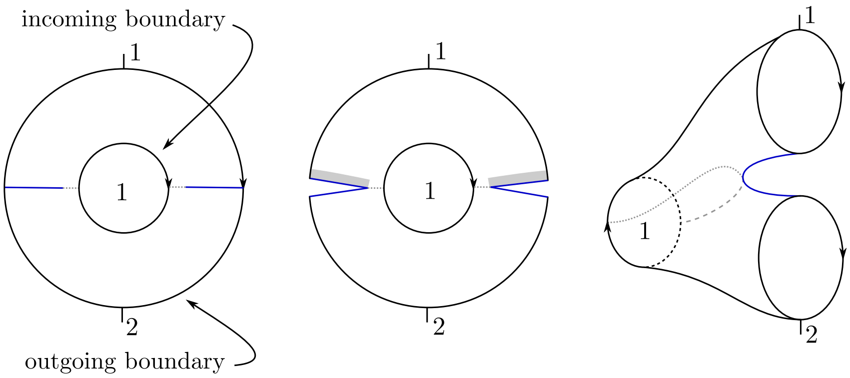

FIGURE 1. An example of constructing a cobordism by cutting and glueing slits in annuli. We start with the annulus on the left, cut along the blue lines to obtain the cut annulus in the middle, and finally glue both the gray sides and the white sides of the cuts to get the cobordism on the right. In this simple example the pairing λ and the successor permutation ω are uniquely determined.

However, there are several problematic situations that could occur. Firstly, if two slits ζ_i and ζ_j lie on the same radial segment (a subset of the annulus $\mathbb{A}^{(j)}_{R_j}$ of the form $\{z \in \mathbb{A}^{(j)}_{R_j} \mid \arg(z) = \theta\}$ for some θ), then our cutting and glueing procedure is not well-defined. We still need to keep track of whether ζ_i lies clockwise or counterclockwise from ζ_j. To do this we also include the data of a successor permutation $\omega : \{1, \ldots, 2h\} \to \{1, \ldots, 2h\}$. This has n cycles, corresponding to the n annuli, and we should demand that each cycle contains the numbers of the slits in one of the annuli and is compatible with the weak cyclic ordering on these coming from the argument of the slits. In a sense the successor permutation keeps track of the fact that when two slits coincide, one lies actually infinitesimally counterclockwise from the other. See Figure 2.

This is still not enough because if all slits on an annulus lie on the same radial segment, we can only deduce the ordering of the slits up to a cyclic permutation. To fix this, we add additional data; the angular distance $r_i \in [0, 2\pi]$ in counterclockwise direction from ζ_i to $\zeta_{\omega(i)}$. In almost all cases one can deduce this from the locations of the ζ_i and ω, but in the case where all slits on an annulus lie on the same radial segment, one of them will have to be $r_i = 2\pi$, while the others will have to be $r_j = 0$. This allows one to determine the ordering of the slits, since the slit ζ_i with $r_i = 2\pi$ should be first in clockwise direction from the angular gap between the slits.

We have almost described enough data to construct a cobordism. We can build a possibly degenerate surface, which has among its boundary components the inner boundaries of the annuli. Since we wanted m outgoing boundary components, we restrict to the subset of data that gives us m boundary components in addition to these inner boundaries of annuli. The inner boundaries of the annuli come with a canonical parametrization, but the outer ones do not have such a parametrization yet. Because they already have a canonical orientation coming from the orientation of the outer boundary of the annuli, it suffices to add one point P_i in each of them, m in total. Finally, we will need to include these new parametrization points in ω and the r_i's. To do this, we write $\xi_i = \zeta_i$ for $1 \leq i \leq 2h$ and $\xi_{2h+i} = P_i$ for $1 \leq i \leq m$, and expand our definition of ω to an permutation $\tilde{\omega}$ of $2h + m$ elements and add additional $r_{2h+i} \in [0, 2\pi]$ for $1 \leq i \leq m$.

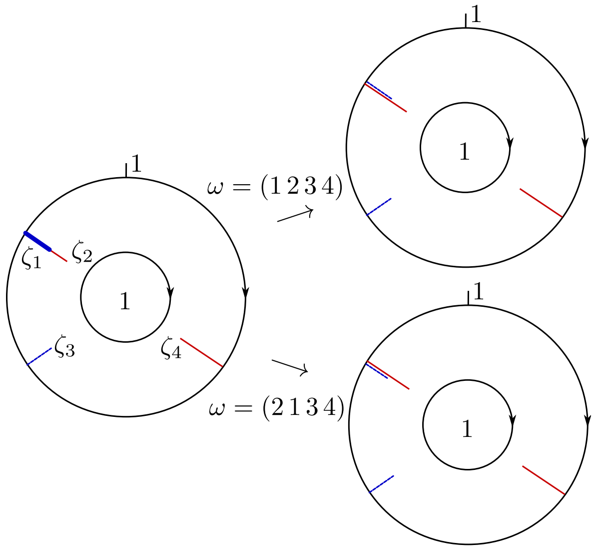

FIGURE 2. An example of a radial slit preconfiguration with a two slits on the same angular segment; ζ_1 is the shorter thick blue slit and ζ_2 is the longer thin red slit. The successor permutation ω allows us to think of ζ_1 as either infinitesimally clockwise or counterclockwise from ζ_2.

Now we make the definition of a radial slit configuration precise. We need to collect all the data, identify configurations leading to the same conformal surface and discard configurations leading to degenerate surfaces. Actually, it is only necessary to consider configurations with a fixed outer radius, we will say more on this towards the end of the section. Therefore, from now on let $\vec{R} = (R, R, \dots, R)$ and $R = \frac{1}{2\pi} + \frac{1}{2}$ unless explicitly stated otherwise. This choice of outer radius is arbitrary, but we have chosen this specific outer radius so that it will become easier to make the connection with metric fat graphs.

Definition 2.1. The space of *possibly degenerate radial slit preconfigurations* $\overline{\mathrm{PRad}}_h(n,m)$ is the subspace of $(\bigsqcup_{j=1}^n \mathbb{C})^{2h} \times \mathfrak{S}_{2h} \times \mathfrak{S}_{2h+m} \times [0,2\pi]^{2h+m} \times (\bigsqcup_{j=1}^n \mathbb{C})^m$ of tuples $L = (\vec{\zeta}, \lambda, \tilde{\omega}, \vec{r}, \vec{P})$ where:

- $\vec{\zeta} \in (\bigsqcup_{j=1}^n \mathbb{C})^{2h}$ are the endpoints of the *slits*
- $\lambda \in \mathfrak{S}_{2h}$ is the *slit pairing*
- $\tilde{\omega} \in \mathfrak{S}_{2h+m}$ is the *successor permutation*
- $\vec{r} \in [0,2\pi]^{2h+m}$ are the *angular distances*
- $\vec{P} \in (\bigsqcup_{j=1}^n \mathbb{C})^m$ are the *parametrization points*

subject to six conditions. For notation, let $\vec{\xi} \in (\bigsqcup_{j=1}^n \mathbb{C})^{2h+m}$ be given by $\xi_i = \zeta_i$ for $1 \leq i \leq 2h$ and $\xi_{i+2h} = P_i$ for $1 \leq i \leq m$ and let $\omega \in \mathfrak{S}_{2h}$ be the restriction of $\tilde{\omega}$ to the set $\{1, 2 \dots 2h\}$. The conditions mentioned are the following:

(i) Each slit ζ_i lies in $\bigsqcup_{j=1}^n \mathbb{A}_R^{(j)} \subset \bigsqcup_{j=1}^n \mathbb{C}$ and each parametrization point P_i lies in $\bigsqcup_{j=1}^n \partial_{out}\mathbb{A}_R^{(j)}$.

(ii) The slit pairing λ consists of h 2-cycles. We demand for all $1 \leq i \leq 2h$ we have that $|\zeta_i| = |\zeta_{\lambda(i)}|$.

(iii) The successor permutation $\tilde{\omega}$ consists of a disjoint union of n cycles and these cycles consist exactly of the indices of the ξ_i lying on each of the annuli. We demand that the permutation action of $\tilde{\omega}$ on these ξ_i preserves the weakly cyclic ordering which comes from the argument (as usual taken in counterclockwise direction).

(iv) The *boundary component permutation* $\lambda \circ \omega$ consists of m cycles. It will turn out that its cycles correspond to the outgoing boundary components.

(v) We demand that P_i lies in the subset O_i of $\bigsqcup_{j=1}^n \partial_{\mathrm{out}}\mathbb{A}_R^{(j)}$ which we will now define. The m cycles of $\lambda \circ \omega$ allow one to write the outer boundaries of the annuli as a union of m subsets, overlapping only in isolated points. We demand that each of these contains exactly one P_i and denote that subset by O_i. To be precise, each O_i is the union of the parts in the outer

boundary between the radial segments ζ_j and $\zeta_{\omega(j)}$ in counter-clockwise direction, for all j in a cycle of $\lambda \circ \omega$.

(vi) The angular distances r_i must be compatible with the location of the ξ_i and the successor permutation $\tilde{\omega}$ in the following sense. If ξ_i does not lie on an annulus with all slits and parametrization points coinciding, then r_i is equal to the angular distance in counterclockwise direction from ξ_i to $\xi_{\tilde{\omega}(i)}$. If ξ_i lies on an annulus with all slits and parametrization points coinciding, then r_i is equal to either 0 or 2π and exactly one ξ_j on that annulus has $r_j = 2\pi$.

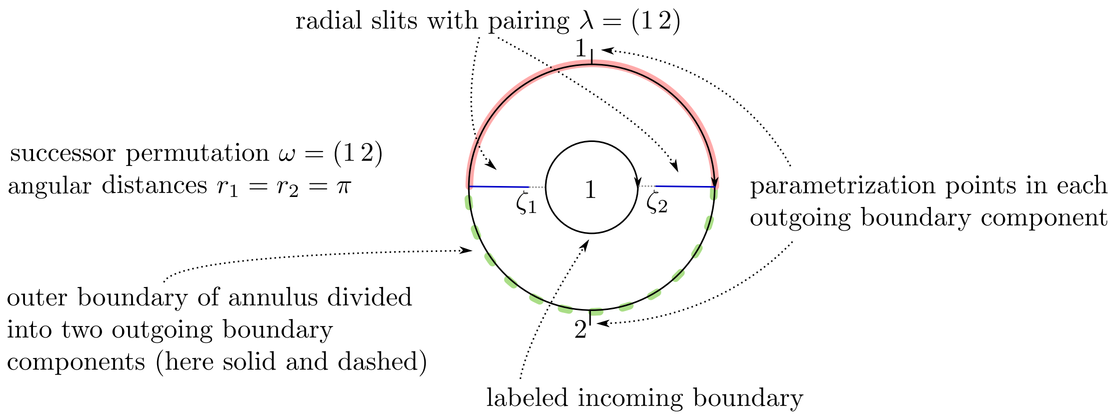

FIGURE 3. The configuration of Figure 1 with all its data pointed out.

Let us now give a precise construction of the possibly degenerate cobordism $S(L)$ obtained out of a preconfiguration L. We first need to define the sector space $\tilde{\Sigma}(L)$, a precise definition of the pieces used in the glueing construction. We slightly depart from our informal discussion by making cuts from the outer boundary to the inner boundary of the annuli and reglueing these later. See Figure 4 for examples of the different types of sectors.

Definition 2.2. Let l be the number of annuli containing no slits. Then $\tilde{\Sigma}(L)$ will have $2h + l$ components. These come in four types:

Ordinary sectors: If $\arg(\zeta_i) \neq \arg(\zeta_{\omega(i)})$ and ζ_i lies on the jth annulus $\mathbb{A}_R^{(j)}$, then we set

$$F_i = \{z \in \mathbb{A}_R^{(j)} \mid \arg(\zeta_i) \leq \arg(z) \leq \arg(\zeta_{\omega(i)})\}$$

Thin sectors: If $\arg(\zeta_i) = \arg(\zeta_{\omega(i)})$, $r_i = 0$ and ζ_i lies on the jth annulus $\mathbb{A}_R^{(j)}$, then we set

$$F_i = \{z \in \mathbb{A}_R^{(j)} \mid \arg(\zeta_i) = \arg(z)\}$$

Full sectors: If $\arg(\zeta_i) = \arg(\zeta_{\omega(i)})$, $r_i = 2\pi$ and ζ_i lies on the jth annulus $\mathbb{A}_R^{(j)}$, then we set F_i to be equal to the annulus $\mathbb{A}_R^{(j)}$ cut open along the segment $\arg(z) = \arg(\zeta_i)$, with that segment doubled so that it is homeomorphic to a closed rectangle.

Entire sectors: If the jth annulus $\mathbb{A}_R^{(j)}$ does not contain any slits and is j'th in the induced ordering on the r annuli that do not contain any slits, we set $F_{2h+j'} = \mathbb{A}_R^{(j)}$.

The surface $\Sigma(L)$ underlying the cobordism $S(L)$ will be obtained as a quotient space of the sector space by an equivalence relation that makes identifications on the boundary of the sectors. We will now define the subsets involved in those identifications.

Definition 2.3. If F_i is an ordinary or thin sector corresponding to the slit ζ_i on the jth annulus $\mathbb{A}_R^{(j)}$, then we define the following subspaces of F_i:

$$\alpha_i^+ = \{z \in \mathbb{A}_R^{(j)} \mid \arg(z) = \arg(\zeta_{\omega(i)}) \text{ and } |z| \leq |\zeta_{\omega(i)}|\}$$
$$\alpha_i^- = \{z \in \mathbb{A}_R^{(j)} \mid \arg(z) = \arg(\zeta_i) \text{ and } |z| \leq |\zeta_i|\}$$
$$\beta_i^+ = \{z \in \mathbb{A}_R^{(j)} \mid \arg(z) = \arg(\zeta_{\omega(i)}) \text{ and } |z| \geq |\zeta_{\omega(i)}|\}$$
$$\beta_i^- = \{z \in \mathbb{A}_R^{(j)} \mid \arg(z) = \arg(\zeta_i) \text{ and } |z| \geq |\zeta_i|\}$$

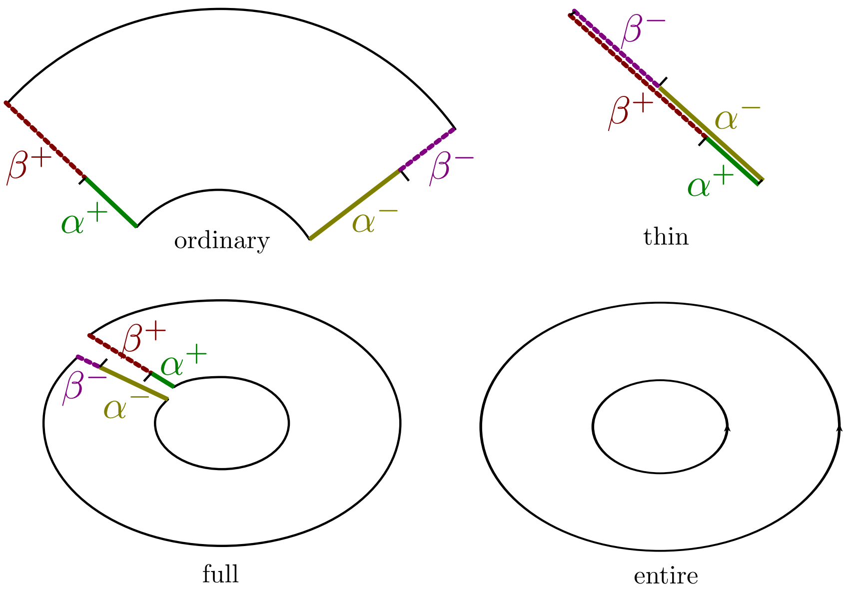

FIGURE 4. Examples of the different types of radial sectors with subsets $\alpha^\pm$ and $\beta^\pm$.

If F_i is a full sector then our definitions have to be slightly different, because now the two radial segments in the boundary have the same argument. Let S_i^+ be the radial segment bounding F_i in counterclockwise direction and S_i^- be the radial segment bounding it in clockwise direction, then we define the following subspaces of F_i:

$$\alpha_i^+ = \{z \in S_i^+ \mid |z| \leq |\zeta_{\omega(i)}|\} \qquad \alpha_i^- = \{z \in S_i^- \mid |z| \leq |\zeta_i|\}$$
$$\beta_i^+ = \{z \in S_i^+ \mid |z| \geq |\zeta_{\omega(i)}|\} \qquad \beta_i^- = \{z \in S_i^- \mid |z| \geq |\zeta_i|\}$$

These subspaces are empty for entire sectors.

We can now define the equivalence relation $\approx_L$ and the surface $\Sigma(L)$.

Definition 2.4. The equivalence relation $\approx_L$ on $\tilde{\Sigma}(L)$ is the one generated by

(i) We identify $z \in \alpha_i^+$ with $z \in \alpha_{\omega(i)}^-$.
(ii) We identify $z \in \beta_i^+$ with $z \in \beta_{\lambda(i)}^-$.

We define the surface $\Sigma(L)$ to be $\tilde{\Sigma}(L)/{\approx_L}$.

Definition 2.5. The cobordism $S(L)$ has underlying surface $\Sigma(L)$. It has a map from each inner boundary $\partial_{\text{in}}\mathbb{A}_R^{(j)}$

$$\iota_j^{\text{in}} : S^1 \cong \partial_{\text{in}}\mathbb{A}_R^{(j)} \to \Sigma(L)$$

which are inclusions of subspaces if none of the slits lie on the inner boundary of an annulus. One can define the outgoing boundary components as a subspace of $\Sigma(L)$ by considering the intersection of the outer boundary of the annuli with the sectors. For each cycle in $\lambda \circ \omega$ these intersections form a circle with canonical orientation and starting point P_k. This gives us for the cycle $\lambda \circ \omega$ corresponding to P_k a map

$$\iota_k^{\text{out}} : S^1 \to \Sigma(L)$$

which are inclusions of subspaces if none of the slits lie on the outer boundary of an annulus.

As mentioned before, this definition might lead to degenerate cobordisms for some L. Moreover, two different pre-configurations might give the same conformal classes of cobordism. In fact, each conformal class of cobordisms occurs at least $(2h)!$ times, because the labeling on the slits does not matter. To see that degenerate surfaces can occur, consider the example in Figure 5. Now we will explain how to resolve both issues.

We have already explained that one might obtain the same surface more than once by permuting the labels on the slits. It turns out we only have to make two additional identifications. For the first additional identification, think instead of doing all the cutting and glueing simultaneously, doing

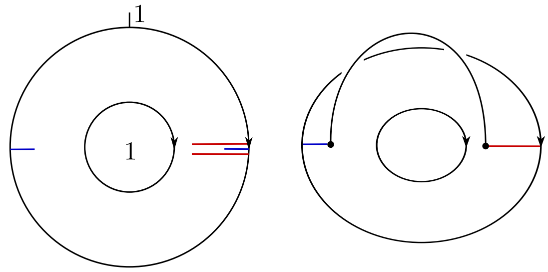

FIGURE 5. An example of a radial slit preconfiguration leading to a degenerate surface. The black arc connecting two points on the surface on the right was the line segment between the two red slits.

it in order of increasing modulus of the slits. This results in the same cobordism but it becomes clear that if ζ_i lies on the same radial segment as ζ_j and satisfies $|\zeta_i| \geq |\zeta_j|$, it might as well be on the other side of $\zeta_{\lambda(j)}$. That is, it might as well have "jumped" over the slit ζ_j to $\zeta_{\lambda(j)}$. For the second additional identification, note that if a parametrization point similarly "jumps" over a slit, this does not change the parametrization of the outgoing boundary. These will turn out to be all required identifications, and we now use them to define equivalence relations on $\mathrm{PRad}_h(n,m)$.

Definition 2.6. Let $\equiv'$ be the equivalence relation on $\overline{\mathrm{PRad}}_h(n,m)$ generated by

Relabeling of the slits: We identify two preconfigurations if they can be obtained from each other by relabeling the slits. More precisely for every permutation $\sigma \in \mathfrak{S}_{2h}$ and $L = (\vec{\zeta}, \lambda, \tilde{\omega}, \vec{r}, \vec{P}) \in \mathrm{PRad}_h(n,m)$ we say that $L \equiv' \sigma(L)$, with

$$\sigma(L) = ((\vec{\zeta})^\sigma, \sigma \circ \lambda \circ \sigma^{-1}, (\tilde{\omega})^\sigma, (\vec{r})^\sigma, \vec{P})$$

with components defined as follows:

- $(\vec{\zeta})^\sigma$ is given by $(\zeta^\sigma)_i = \zeta_{\sigma(i)}$,
- $(\vec{r})^\sigma$ is given by $(r^\sigma)_i = r_{\tilde{\sigma}(i)}$, where $\tilde{\sigma} \in \mathfrak{S}_{2h+m}$ is the permutation induced by extending σ by the identity,
- $(\tilde{\omega})^\sigma = \tilde{\sigma} \circ \tilde{\omega} \circ \tilde{\sigma}^{-1}$.

Let $\equiv$ be the equivalence relation on $\overline{\mathrm{PRad}}_h(n,m)$ generated by relabeling of the slits (as above) and the following two identifications:

Slit jumps: We say $L \equiv L'$ if L' can be obtained from L by a slit jump, see Figure 6. More precisely, if we are given a preconfiguration L and two indices i and j such that $j = \omega(i)$, $r_i = 0$ and $|\zeta_i| \geq |\zeta_j|$, then we can obtain a new preconfiguration L' as follows. We replace ζ_i by the point $\zeta_i' = \frac{|\zeta_i|}{|\zeta_{\lambda(j)}|}\zeta_{\lambda(j)}$ and keep all the other slits the same. We then put i after of $\lambda(j)$ in $\tilde{\omega}$ to obtain $\tilde{\omega}'$ and set $r_i' = r_{\lambda(j)}$ and $r'_{\lambda(j)} = 0$. The rest of the data remains the same.

Parametrization point jumps: We say $L \equiv L'$ if L' can be obtained from L by a jump of a parametrization point, see Figure 7. More precisely, if we are given a preconfiguration L in which there is a P_i such that $j = \tilde{\omega}(i + 2h)$ for some j and $r_{i+2h} = 0$, then we can obtain a new preconfiguration L' by keeping all the data the same except replacing P_i with P_i' lying at the radial segment through $\zeta_{\lambda(j)}$ and setting $r'_{i+2h} = r_{\lambda(j)}$ and $r'_{\lambda(j)} = 0$.

Definition 2.7. We now define certain quotient spaces using these equivalence relations.

- The space $\overline{\mathrm{QRad}}_h(n,m)$ of *unlabeled possibly degenerate radial slit configurations* is the quotient of $\overline{\mathrm{PRad}}_h(n,m)$ by $\equiv'$.
- The space $\overline{\mathrm{Rad}}_h(n,m)$ of *possibly degenerate radial slit configurations* is the quotient of $\overline{\mathrm{PRad}}_h(n,m)$ by $\equiv$.

We will denote by $[L]$ the radial slit configuration represented by a preconfiguration L. We are left to deal with the problem that certain preconfigurations give cobordisms whose underlying surface is degenerate. We call such preconfigurations *degenerate*. In [Böd06], Bödigheimer gave a necessary and sufficient criterion for a (pre)configuration to lead to a degenerate surface. We state his result now.

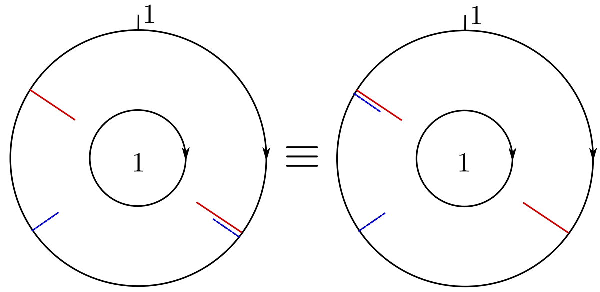

FIGURE 6. A jump of a slit. The pairing λ is given by the colors, but is uniquely determined by the configuration.

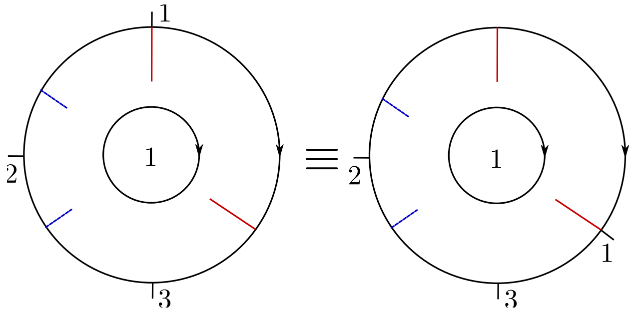

FIGURE 7. A jump of a parametrization point.

Proposition 2.8. *The surface underlying the cobordism $\Sigma(L)$ constructed out of a preconfiguration L is degenerate if and only if it is equivalent under $\equiv$ to a preconfiguration satisfying at least one of the following three conditions:*

Slit hitting inner boundary: *There is a slit ζ_i with $|\zeta_i| = \frac{1}{2\pi}$.*

Slit hitting outer boundary: *There is a slit ζ_i on an annulus $\mathbb{A}_R^{(j)}$ with $|\zeta_i| = R_j$.*

Slits are "squeezed": *There is a pair i, j such that $j = \lambda(i)$, ζ_i and ζ_j lie on the same annulus, $\zeta_i = \zeta_j$ and such that for all k between i and j in the cyclic ordering coming from ω, we have that $|\zeta_k| \geq |\zeta_i| = |\zeta_j|$ (see Figure 5 for an example). If all slits on the annulus containing ζ_i and ζ_j lie at the same point, we additionally require that $r_k = 0$ for all of the k between i and j.*

Definition 2.9. A radial slit preconfiguration is said to be *generic* if it is not equivalent to any other by slit or parametrization point jumps, i.e. all the slits are disjoint.

Definition 2.10. We define the following spaces:

- The space $\mathrm{PRad}_h(n,m)$ of *unlabeled radial slit configurations* is the subspace of $\overline{\mathrm{PRad}}_h(n,m)$ consisting of non-degenerate preconfigurations.
- The space $\mathrm{QRad}_h(n,m)$ of *unlabeled radial slit configurations* is the subspace of $\overline{\mathrm{QRad}}_h(n,m)$ consisting of equivalence classes with non-degenerate representatives.
- The space $\mathrm{Rad}_h(n,m)$ of *radial slit configurations* is the subspace of $\overline{\mathrm{Rad}}_h(n,m)$ consisting of equivalence classes with non-degenerate representatives.

2.1.2. *Cell complexes of radial slit configurations.* Next we give CW complexes $\overline{\mathfrak{Rad}}$ and $\mathfrak{Rad}$ homeomorphic to the spaces of radial slit configurations given before. On $\overline{\mathfrak{Rad}}$ this is the CW structure given in Section 8.2 of [Böd06] and on the subspace $\mathfrak{Rad}$ it coincides with the radial analogue of [Böd07]. The cells will be indexed by so-called combinatorial types, which we define first.

Definition 2.11. Fix an L in $\mathrm{PRad}_h(n,m)$.

- The radial segments of the slits, the parametrization points and the positive real lines, divide the annuli of the preconfiguration L radially into different pieces, which we will call *radial chambers* (see Figure 8).

- Each slit ζ_i in L defines a circle of radius $|\zeta_i|$ on all of the n annuli. These circles divide the n annuli into different pieces, which we will call *annular chambers* (see Figure 8).

Remark 2.12. The orientation of the complex plane endows the radial chambers on each annulus with a natural ordering, and similarly the modulus endows the annular chambers with a natural ordering (see Figure 8).

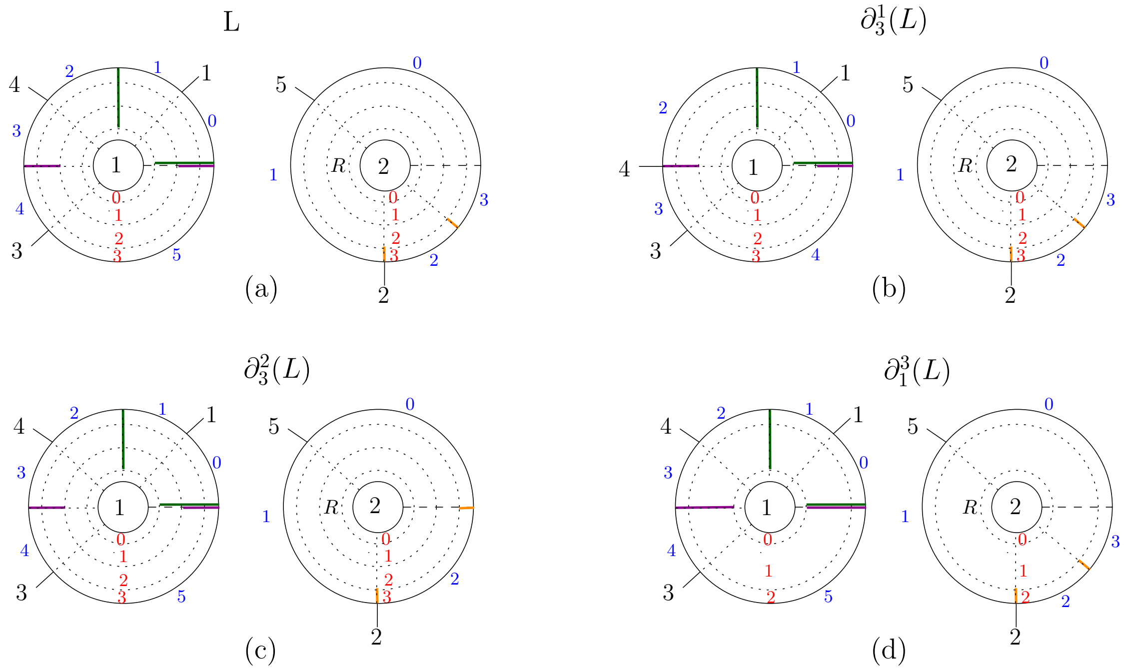

FIGURE 8. (a) A configuration L and its radial and annular chambers divided by dotted lines. The radial chambers are numbered in blue and the annular chambers are numbered in red. This combinatorial type gives an 11-cell in $\overline{\mathfrak{Rad}}$ given by a $\Delta^5 \times \Delta^3 \times \Delta^3$. (b), (c), (d) show part of the boundary of L and their chambers.

Thus annular chambers are always homeomorphic to a disjoint union of annuli, while radial chambers are always homeomorphic to rectangles.

Definition 2.13. Two preconfigurations L and L' in $\mathrm{PRad}_h(n,m)$ are said to *have the same combinatorial data* if L' can be obtained from L by continuously moving the slits and parametrization points in each complex plane without collapsing any chamber. This defines an equivalence relation on $\mathrm{PRad}_h(n,m)$.

A *combinatorial type of preconfigurations* $\mathcal{L}$ is an equivalence class of preconfigurations under this relation. Informally, a combinatorial type is the data carried over by the picture of a preconfiguration without remembering the precise placement of the slits. Notice that this equivalence relation is also well defined on the sets of radial slit configurations $[L]$. Thus one can similarly define a *combinatorial type of configurations* $[\mathcal{L}]$ to be an equivalence class of configurations under this relation. Similarly for the case of unlabeled radial slit configurations.

We will use Υ for the *set of all combinatorial types of configurations*.

Remark 2.14. Notice that if L is a degenerate (respectively non-degenerate) preconfiguration then so is any preconfiguration of the same combinatorial type. Thus, we can talk about a degenerate or non-degenerate combinatorial type.

Remark 2.15. In [Böd07], Bödigheimer gives an elegant definition of a combinatorial type of a (non-degenerate) radial slit configuration in terms of tuples of elements of symmetric groups. The main advantage of that description is that it does not need the notion of a preconfiguration or equivalence relations between them. Unfortunately, we can not use it in this paper since this definition does not extend to the degenerate configurations as they have been defined in this paper.

Now we give definitions of cell complexes of (pre)configurations and their compactifications. Note that the meaning of p and q is different from [Böd06].

Definition 2.16. The *multi-degree of a combinatorial type* $[\mathcal{L}]$ on n annuli is the $(n+1)$-tuple of integers $(q_1, \ldots, q_n, p)$ where $q_i + 1$ is the number of radial chambers in the ith annulus and $p+1$ is the number of annular chambers. For $0 \leq j \leq q_i$ and $0 \leq i \leq n$, we denote by $d^i_j([\mathcal{L}])$, the combinatorial type obtained by collapsing the jth radial chamber on the ith annulus, see Figure 8. For $0 \leq j \leq p$, we denote by $d^{n+1}_j([\mathcal{L}])$, the combinatorial type obtained by collapsing the jth annular chamber, see Figure 8.

The *cell complex of possibly degenerate radial slit configurations* $\overline{\mathfrak{Rad}}_h(n,m)$ is the realization of the multisimplicial set with:

- $(q_1, \ldots, q_n, p)-$simplicies given by $\{\sigma_{[\mathcal{L}]} | [\mathcal{L}]$ combinatorial type of multi-degree $(q_1, \ldots, q_n, p)\}$,
- the faces of $\sigma_{[\mathcal{L}]}$ given by $d^i_j(\sigma_{[\mathcal{L}]}) := \sigma_{d^i_j([\mathcal{L}])}$.

That is, $\overline{\mathfrak{Rad}}_h(n,m)$ is a CW-complex with cells indexed by combinatorial types of radial slits configurations as follows. Let $e_{[\mathcal{L}]} := \Delta^{q_1} \times \ldots \times \Delta^{q_n} \times \Delta^p$, then:

$$\overline{\mathfrak{Rad}}_h(n,m) := \frac{\bigsqcup_{[\mathcal{L}] \in \Upsilon} e_{[\mathcal{L}]}}{\sim}$$

where the equivalence relation is generated by

$$(e_{[\mathcal{L}]}, (\vec{t}_1, \ldots, \delta^j(\vec{t}_i), \ldots, \vec{t}_{n+1})) \sim (e_{d^i_j([\mathcal{L}])}, (\vec{t}_1, \ldots, \vec{t}_i, \ldots, \vec{t}_{n+1}))$$

where δ^j is the map $\Delta^{q_i - 1} \to \Delta^{q_i}$ including 0 as the $(j+1)$st coordinate, and Υ is the set of combinatorial types of radial slit configurations.

The cell complexes of possibly degenerate radial slit preconfigurations $\overline{\mathfrak{PRad}}_h(n,m)$ and unlabeled configurations $\overline{\mathfrak{QRad}}_h(n,m)$ are defined in similar ways.

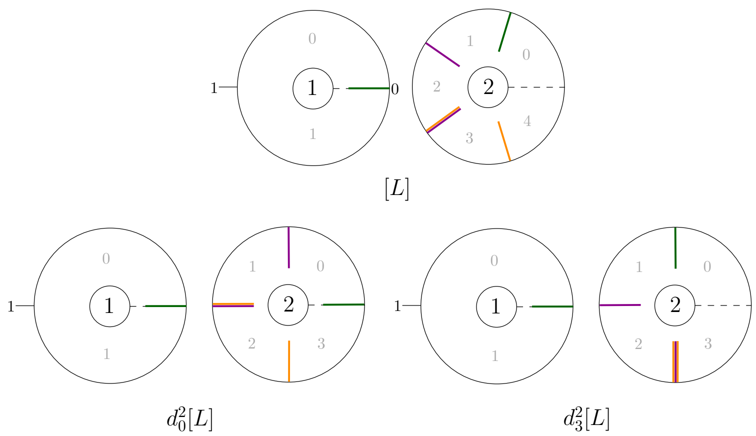

FIGURE 9. A 5-cell given by $\Delta^1 \times \Delta^4$ in $\overline{\mathfrak{Rad}}$ and part of its boundary. The radial chambers are numbered in grey.

Definition 2.17. If a combinatorial type $[\mathcal{L}]$ is degenerate, then $d^i_j([\mathcal{L}])$ is also degenerate. Thus, we define the *cell complex of degenerate radial slit configurations* as the subcomplex $\overline{\mathfrak{Rad}}_h(n,m)' \subset \overline{\mathfrak{Rad}}_h(n,m)$ obtained as the realization of the degenerate simplices. Finally, the $\mathfrak{Rad}_h(n,m)$ is the complement. That is

$$\mathfrak{Rad}_h(n,m) := \overline{\mathfrak{Rad}}_h(n,m) - \overline{\mathfrak{Rad}}_h(n,m)'$$

The spaces $\mathfrak{PRad}_h(n,m)$ and $\mathfrak{QRad}_h(n,m)$ are defined in a similar way.

We introduce notation for the image of $e_{[\mathcal{L}]}$ in $\mathfrak{Rad}$.

Definition 2.18. Let $[\mathcal{L}]$ be a combinatorial type, we define the subspace $\mathfrak{Rad}_{[\mathcal{L}]}$ as image of the interior of $e_{[\mathcal{L}]}$. We also let $\overline{\mathfrak{Rad}}_{[\mathcal{L}]}$ be the closure of $\mathfrak{Rad}_{[\mathcal{L}]}$ in $\overline{\mathfrak{Rad}}$ and define $\partial\overline{\mathfrak{Rad}}_{[\mathcal{L}]} = \mathfrak{Rad} \cap (\overline{\mathfrak{Rad}}_{[\mathcal{L}]} \backslash \mathfrak{Rad}_{[\mathcal{L}]})$.

2.1.3. *Relationships.* Our final goal for this section is to explain the relationship between the spaces and cell complexes of radial slit configurations, and the moduli space of cobordisms. The first relationship is straightforward, as there are obvious continuous bijections

$$\begin{aligned} \mathrm{Rad}_h(n,m) &\to \mathfrak{Rad}_h(n,m) & \overline{\mathrm{Rad}}_h(n,m) &\to \overline{\mathfrak{Rad}}_h(n,m) \\ \mathrm{QRad}_h(n,m) &\to \mathfrak{QRad}_h(n,m) & \overline{\mathrm{QRad}}_h(n,m) &\to \overline{\mathfrak{QRad}}_h(n,m) \\ \mathrm{PRad}_h(n,m) &\to \mathfrak{PRad}_h(n,m) & \overline{\mathrm{PRad}}_h(n,m) &\to \overline{\mathfrak{PRad}}_h(n,m) \end{aligned}$$

compatible with the quotient maps and inclusions. These are given by sending a point to its combinatorial type and the simplicial coordinates obtained by rescaling the angles of the slits (for the first n coordinates) and their radii (for the last coordinate). The following Lemma follows from [Böd06] and we sketch a proof below.

Lemma 2.19. *These maps are homeomorphisms.*

Proof. We start by noting that $\overline{\mathrm{PRad}}_h(n,m)$ and $\overline{\mathfrak{PRad}}_h(n,m)$ are both compact Hausdorff spaces; the former is a closed subset of a compact Hausdorff space and the latter is a finite CW-complex. A continuous bijection between compact Hausdorff spaces is a homeomorphism. Next note that the maps $\overline{\mathrm{Rad}}_h(n,m) \to \overline{\mathfrak{Rad}}_h(n,m)$ and $\overline{\mathrm{QRad}}_h(n,m) \to \overline{\mathfrak{QRad}}_h(n,m)$ are induced by passing to quotients, as are their inverses, so they are also homeomorphisms.

Thus the right maps are homeomorphisms and the left maps are obtained by restricting these homeomorphisms to open subsets and replacing their codomain with their image. Hence they are also homeomorphisms. □

The relationship to moduli space is less straightforward. In Section 9 of [Böd06] Bödigheimer defined a space $\overline{\mathbf{RAD}}_h(n,m)$ of all radial slit configurations with varying inner radii, but fixed outer radii and a subspace $\mathbf{RAD}_h(n,m)$ of all non-degenerate radial slit configurations. He also proved a version of the previous Lemma.

Lemma 2.20. *We have homotopy equivalences*

$$\mathbf{RAD}_h(n,m) \simeq \mathrm{Rad}_h(n,m) \qquad \overline{\mathbf{RAD}}_h(n,m) \simeq \overline{\mathrm{Rad}}_h(n,m)$$

Sketch of proof. To explain the existence of these homotopy equivalences, we note that Bödigheimer's $\overline{\mathbf{RAD}}$ and $\mathbf{RAD}$ differ from $\overline{\mathrm{Rad}}$ and Rad only in the following two ways:

(i) In $\overline{\mathbf{RAD}}$ and $\mathbf{RAD}$, the inner radii are allowed to vary in $(0, R_0)$ for some choice of $R_0 > 0$, while in $\overline{\mathrm{Rad}}$ and Rad they are fixed to $\frac{1}{2\pi}$.
(ii) In $\overline{\mathbf{RAD}}$ and $\mathbf{RAD}$, an exceptional set Ω is used to remove ambiguity when all slits on an annulus lie on two segments, while in $\overline{\mathrm{Rad}}$ and Rad this role is played by the angular distances $\vec{r}$.

The second of these encodes equivalent data: given the rest of the data of a radial slit configuration, Ω can be reconstructed from $\vec{r}$ and vice versa. The first tells us that the difference between the two spaces is the contractible space of choices of radii. More precisely, there is an inclusion $\overline{\mathrm{Rad}} \hookrightarrow \overline{\mathbf{RAD}}$ with a homotopy inverse given by decreasing all radii to $\min(R_i)$ and changing the radial coordinates of all the data by an affine transformation that sends $\min(R_i)$ to $\frac{1}{2\pi}$ and fixes 1. This homotopy equivalence restricts to one between $\mathbf{RAD}$ and Rad. □

Bödigheimer proved in section 7.5 of [Böd06], with additional details in [Ebe03], that a version of $\mathbf{RAD}_h(n,m)$ without parametrization points on the outgoing boundary, is a model for the moduli space of cobordisms without parametrization of the outgoing boundary.. This uses that the cobordism $\Sigma(L)$ comes with a canonical conformal structure, being obtained by gluing subsets of $\mathbb{C}$. Adding in the parametrizations for the outer boundary, this result easily implies the following theorem:

Theorem 2.21 (Bödigheimer)**.** *The map that assigns to each $[L] \in \mathbf{RAD}_h(n,m)$ the conformal class of the cobordism $S(L)$ gives us a homeomorphism*

$$\mathbf{RAD}_h(n,m) \cong \bigsqcup \mathcal{M}_g(n,m)$$

where the disjoint union is over all isomorphism classes of two-dimensional cobordisms with n incoming boundary components, m outgoing boundary components and total genus g determined by $h = 2g - 2 + n + m$. By the remarks above we have that

$$\mathfrak{Rad}_h(n,m) \simeq \bigsqcup_{[\Sigma]} B\mathrm{Diff}(\Sigma, \partial\Sigma)$$

where the disjoint union is over the same isomorphism classes of surfaces as above.

Bödigheimer actually proved this for connected cobordisms with no parametrization of the outgoing boundary, but this version of the theorem is an easy consequence of his. His proof amounts to checking that $\mathbf{RAD}_h(n,m)$ is a manifold of dimension $3h+m+n$ (see also [EF06] for remarks on the real-analytic structure). It sits as a dense open subset in $\overline{\mathbf{RAD}}_h(n,m)$. In this way we can think of $\overline{\mathbf{RAD}}_h(n,m)$ as a “compactification” of $\mathbf{RAD}_h(n,m)$. Geometrically, one can think of it as the compactification where handles or boundary components can degenerate to radius zero, as long as there is always a path from each incoming to an outgoing boundary component that does not pass through any degenerate handles or boundary components. Colloquially, “the water must always be able to leave the tap.” Bödigheimer calls this the *harmonic compactification of moduli space*. We now describe a deformation retract of the harmonic compactification.

Definition 2.22. The *unilevel harmonic compactification* $\overline{\mathfrak{URad}}_h(n,m)$ is the subspace of $\overline{\mathfrak{Rad}}_h(n,m)$ given by cells corresponding to configurations satisfying $|\zeta_i| = R$ for all $i \in \{1,\ldots,2h\}$, i.e. all slits lie on the outer radius.

Note that in addition to the inclusion $\iota : \overline{\mathfrak{URad}}_h(n,m) \hookrightarrow \overline{\mathfrak{Rad}}_h(n,m)$, there is also a projection $p : \overline{\mathfrak{Rad}}_h(n,m) \to \overline{\mathfrak{URad}}_h(n,m)$ which makes all slits have modulus R. This is a homotopy inverse to the inclusion.

Lemma 2.23. *The maps ι and p are mutually inverse up to homotopy.*

Proof. First note that $p \circ \iota$ is equal to the identity on $\overline{\mathfrak{URad}}$. For $\iota \circ p$, note that a homotopy from the identity on $\overline{\mathfrak{Rad}}$ to $\iota \circ p$ is given at time $t \in [0,1]$ sending each slit ζ_i to $\frac{(1-t)|\zeta_i| + Rt}{|\zeta_i|}\zeta_i$ under the homeomorphism with Rad. $\square$

The spaces constructed in this section fit together in the following diagram

$$\begin{tikzcd}
\mathfrak{PRad}_h(n,m) \arrow[r, "\text{compactification}"] \arrow[d] & \overline{\mathfrak{PRad}}_h(n,m) \arrow[d] & \\
\mathfrak{QRad}_h(n,m) \arrow[r, "\text{compactification}"] \arrow[d] & \overline{\mathfrak{QRad}}_h(n,m) \arrow[d] & \\
\mathfrak{Rad}_h(n,m) \arrow[r, "\text{compactification}"] & \overline{\mathfrak{Rad}}_h(n,m) \arrow[r, shift left, "\simeq"] & \overline{\mathfrak{URad}}_h(n,m) \arrow[l, shift left]
\end{tikzcd}$$

where all the horizontal maps are surjections.

Remark 2.24. One can make sense of glueing of cobordisms on the level of radial slits, see [Böd06]. This construction gives $\mathbf{RAD}_h(n,m)$ the structure of a prop in topological spaces. One of the advantages of the radial slit configurations over fat graphs is the ease with which one can describe the prop structure.

2.2. **The universal surface bundle.** In the previous section, we motivated the definition of the space of radial slit configurations by explaining how each preconfiguration consists of data to a cobordism $S(L)$, and the choice of topology on the radial slit configurations was guided by the idea that this construction should produce conformal families of cobordisms. In this section we will make this precise by defining a universal surface bundle over $\mathfrak{Rad}$ via its homeomorphism with Rad.

Note that the equivalence relation $\equiv$ on $\mathrm{PRad}_h(n,m)$ has the property that there is a canonical isomorphism of cobordisms with conformal structure between $S(L)$ and $S(L')$ if $L \equiv L'$. This allows us to make sense of the cobordism $S([L])$ for an equivalence class $[L]$. The idea for constructing the universal surface bundle over $\mathrm{Rad}_h(n,m)$, is to make the construction of $S([L])$ continuous in $[L]$.

The result is a space over $\mathrm{Rad}_h(n,m)$ and we check it is a bundle and universal by comparing it to the definition of the universal bundle in conformal construction of moduli space.

We start by making sense of the radial sectors $\tilde{\Sigma}(L)$ as a space over $\overline{\mathrm{PRad}}_h(n,m)$. This definition seems obvious; we think of the sectors as a subspace of a disjoint union of annuli for each L, so one is tempted to just state that $\tilde{\Sigma}(L)$ is the relevant subspace of $\overline{\mathrm{PRad}}_h(n,m) \times \left(\bigsqcup_{j=1}^n \mathbb{A}_R^{(j)}\right)$. There are two problems with this: (i) the full sectors are not actually subspaces of annuli and (ii) the number of entire sectors is not constant over $\overline{\mathrm{PRad}}_h(n,m)$.

Both problems are relatively harmless. Then second is solved by noting that the number of entire sectors is locally constant, so one can work separately over each of the subspaces of components with a fixed number of entire sectors. The first is a bit harder to solve, but one deals with it by considering a version of $\overline{\mathrm{PRad}}_h(n,m)$ where the preconfigurations L are endowed with lifts of the slits to elements of $\bigsqcup_{i=1}^n \tilde{\mathbb{A}}_R$, the disjoint union of the universal covers of the annuli, under the condition that the distances between them are still equal to the angular distances. Over this version one has a space with fibers given by $\bigsqcup_{i=1}^n \tilde{\mathbb{A}}_R$, which does contain the full sectors. One then notes that there is a canonical homeomorphism between the sectors over the same configurations with different choices of lifts. In the end, the conclusion is that there exists a space $\tilde{\mathbb{A}}$ over $\overline{\mathrm{PRad}}_h(n,m)$ whose fibers consist of a disjoint union of annuli, and there is a subspace $\overline{\mathrm{PS}}_h(n,m) \subset \tilde{\mathbb{A}}$ whose fiber over L can be canonically identified with the sector space $\tilde{\Sigma}(L)$.

Recall that $\approx_L$ is the equivalence relation on $\tilde{\Sigma}(L)$ used when glueing the sectors together to obtain a surface. Using it fiberwise defines an equivalence relation $\sim$:

Definition 2.25. Let $\sim$ be the equivalence relation on $\overline{\mathrm{PS}}_h(n,m)$ generated by $(L,z) \sim (L',z')$, where $L, L' \in \overline{\mathrm{PRad}}_h(n,m)$, $z \in \tilde{\Sigma}(L) \subset \overline{\mathrm{PS}}_h(n,m)$ and $z' \in \tilde{\Sigma}(L') \subset \overline{\mathrm{PS}}_h(n,m)$, if $L = L'$ and $z \approx_L z'$.

As mentioned before, there is a canonical isomorphism $\phi_{L,L'}$ between $\Sigma(L)$ and $\Sigma(L')$ if $L \equiv L'$. Using this we can define a version of $\equiv$ for $\overline{\mathrm{PS}}_h(n,m)$.

Definition 2.26. Let $\cong$ be the equivalence relation on $\overline{\mathrm{PS}}_h(n,m)$ generated by $\sim$ and by saying that (L,z) and (L',z') are equivalent if $L \equiv L'$ and $z' = \phi_{L,L'}(z)$.

We can now define the surface bundle.

Definition 2.27. We define $\mathrm{PS}_h(n,m)$ to be the restriction of $\overline{\mathrm{PS}}_h(n,m)$ to $\mathrm{Rad}_h(n,m)$. We then define $\mathrm{S}_h(n,m)$ as $\mathrm{PS}_h(n,m)/{\cong}$, which is a space over $\mathrm{Rad}_h(n,m)$.

A priori this is a space over $\mathrm{Rad}_h(n,m)$ with fibers having the structure of cobordisms, but it is in fact a universal surface bundle. This is implicit in [Böd06] but not explicitly stated there. We explain the reasoning below:

Proposition 2.28. *The space* $\mathrm{S}_h(n,m)$ *over* $\mathrm{Rad}_h(n,m)$ *is a universal surface bundle.*

Sketch of proof. Varying radii allows one to extend $\mathrm{S}_h(n,m)$ to $\mathbf{RAD}_h(n,m)$. Theorem 2.21 tells us that the assignment $[L] \mapsto [S([L])]$ is a homeomorphism $\mathbf{RAD}_h(n,m) \to \mathcal{M}_g(n,m)$. Pulling back the universal bundle over $\mathcal{M}_g(n,m)$ defined at the end of Subsection 1.1 exactly gives $\mathrm{S}_h(n,m)$. □

There is a universal $\mathrm{Mod}(S_{g,n+m})$-bundle over $\mathrm{Rad}_h(n,m)$ given by the bundle with fiber over $[L]$ the isotopy classes diffeomorphisms of $\Sigma(L)$ fixing the boundary. We give an alternative explicit construction of this bundle in Definition 4.23.

3. Admissible fat graphs and string diagrams

3.1. **The definition.** Following the ideas of Strebel [Str84], Penner, Bowditch and Epstein gave a triangulation of Teichmüller space of surfaces with decorations, which is equivariant under the action of its corresponding mapping class group [Pen87, BE88]. In this triangulation, simplices correspond to equivalence classes of marked fat graphs and the quotient of this triangulation gives a combinatorial model of the moduli space of surfaces with decorations. These concepts were studied by Harer for the case of surfaces with punctures and boundary components [Har86]. These ideas where later used by Igusa to construct a category of fat graphs that models the mapping class groups of punctured surfaces [Igu02]. Godin extended Igusa's construction for the cases of surfaces with boundary and for open-closed cobordisms [God07b, God07a].

In this section we will define a category of fat graphs and specific subcategories of it in the spirit of Godin. We also define the space of metric fat graphs in the spirit of Harer and Penner, and specific subspaces of these spaces. We then show that they are the classifying spaces of these categories. At the end of the section we define the space of Sullivan diagrams. This is a quotient of a certain subspace of the space of metric fat graphs, and plays the role of a compactification.

3.1.1. *Fat graphs.* We start with precise definitions of graphs and fat graphs.

Definition 3.1. A *combinatorial graph* G is a tuple $G = (V, H, s, i)$, with a finite set of *vertices* V, a finite set of *half edges* H, a *source map* $s : H \to V$ and an *edge pairing* involution $i : H \to H$ without fixed points .

The source map s ties each half edge to its source vertex, and the edge pairing involution i attaches half edges together. The set E of *edges* of the graph is the set of orbits of i. The *valence* of a vertex $v \in V$ is the cardinality of the set $s^{-1}(v)$. A *leaf* of a graph is a univalent vertex and an *inner vertex* is a vertex that is not a leaf. The *geometric realization* of a combinatorial graph G is the CW-complex $|G|$ with one 0-cell for each vertex, one 1-cell for each edge and attaching maps given by s and $s \circ i$. A *tree* is a graph whose geometric realization is a contractible space and a *forest* is a disjoint union of trees.

Definition 3.2. A *fat graph* $\Gamma = (G, \sigma)$ is a combinatorial graph together with a cyclic ordering σ_v of the half edges incident at each vertex v. The *fat structure* of the graph is given by the data $\sigma = (\sigma_v)$ which is a permutation of the half edges.

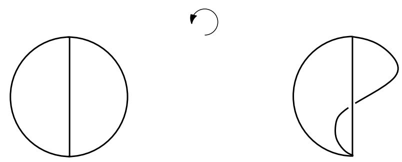

Figure 10. Two different fat graphs – where the fat structure is given by the orientation of the plane, here denoted by the circular arrow – with the same underlying combinatorial graph.

From a fat graph $\Gamma = (G, \sigma)$ one can construct a surface with boundary Σ_Γ by thickening the edges and the vertices. More explicitly, one can construct this surface by replacing each edge with a strip and glueing these strips to a disk at each vertex according to the fat structure. The cyclic ordering exactly gives the data required to do this. Notice that there is a strong deformation retraction of Σ_Γ onto $|G|$ so one can think of $|G|$ as the skeleton of the surface.

Definition 3.3. The *boundary cycles* of a fat graph are the cycles of the permutation of half edges given by $\omega = \sigma \circ i$. Each cycle τ of ω gives a list of edges of the graph Γ and thus determines a subgraph $\Gamma_\tau \subset \Gamma$, which we call the *boundary graph* corresponding to τ.

Remark 3.4. Note that the fat structure of Γ is completely determined by ω. Moreover, one can show that the boundary cycles of a fat graph $\Gamma = (G, \omega)$ correspond to the boundary components of Σ_Γ (cf. [God07b]). Therefore, the surface Σ_Γ is completely determined up to topological type by the combinatorial graph and its fat structure.

A fat graph gives one a surface, but not yet a cobordism. The difference is that it does not distinguish between incoming and outgoing boundary components, nor do these come with canonical parametrizations. Note that after deciding whether a boundary component is incoming or outgoing, a parametrization is uniquely determined once we pick a marked point. Thus it suffices to add to each boundary component a leaf labeled either “incoming” or “outgoing.”

Definition 3.5. A *closed fat graph* $\Gamma = (\Gamma, L_{\text{in}}, L_{\text{out}})$ is a fat graph with an ordered set of leaves and a partition of this set of leaves into two sets L_{in} and L_{out}, such that:

(i) all inner vertices are at least trivalent,
(ii) there is exactly one leaf on each boundary cycle. Given a leaf l_i we denote its corresponding boundary graph by $\Gamma_{l_i} \subset \Gamma$.

Leafs in L_{in} or in L_{out}, are called *incoming* or *outgoing* respectively.

Note that the previous definition also removed unnecessary bivalent and univalent vertices. It turns out that one can consider an even more restricted type of fat graph, which reflects that (like in radial slits) we can decide to arrange the incoming boundary in a special way.

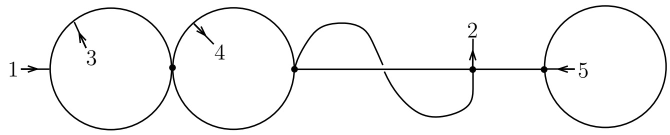

FIGURE 11. An example of a closed fat graph which is not admissible. The incoming and outgoing leaves are marked by incoming or outgoing arrows.

Definition 3.6. Let Γ be a closed fat graph. Let l_i denote a leave of Γ and $\Gamma_{l_i} \subset \Gamma$ be its corresponding boundary graph. Γ is called *admissible* if the subgraphs $\Gamma_{l_i} - l_i$ for all incoming leaves l_i are disjoint embedded circles in Γ. We refer to these boundary cycles as *admissible cycles* (see Figure 12).

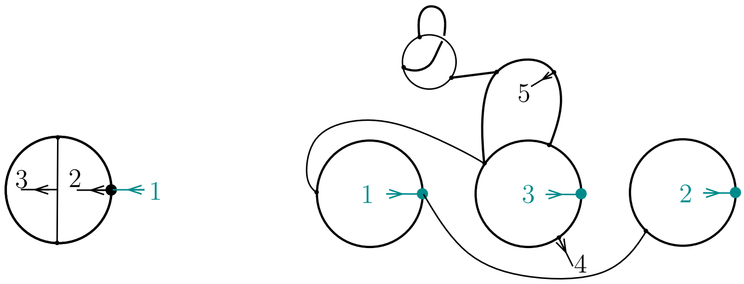

FIGURE 12. Two examples of admissible fat graphs. The graph on the left has the topological type of the pair of pants and the one on the right of a surface of genus 1 with 5 boundary components.

We organize fat graphs into a category. The idea is that when we use fat graphs to construct surfaces, we should be able to pick different lengths for the edges to obtain different conformal classes. Furthermore, if the length of an edge goes to zero, we expect the two disks corresponding to the vertices to be glued together. This makes sense as long as the edge is not a loop. The morphisms in the category of fat graphs encode this relationship between graphs. Recall that a tree is a graph whose geometric realization is contractible and a forest is a disjoint union of trees.

Definition 3.7. We define two categories:

- The *category of closed fat graphs* $\mathcal{F}at$ is the category with objects isomorphism classes of closed fat graphs and morphisms $[\Gamma] \to [\Gamma/F]$ given by collapses of subforests of Γ which do not contain any leaves.
- The *category of admissible fat graphs* $\mathcal{F}at^{ad}$ is the full subcategory of $\mathcal{F}at$ with objects isomorphism classes of admissible fat graphs.

The composition in $\mathcal{F}at$ and $\mathcal{F}at^{ad}$ and hence the categories themselves, are well defined. The category $\mathcal{F}at$ was introduced by Godin in [God07b] and $\mathcal{F}at^{ad}$ is a slight variation of it introduced by the same author in [God07a].

Note that the collapse of a subforest which does not contain any leaves induces a surjective homotopy equivalence upon geometric realizations and does not change the number of boundary components. Therefore, if there is a morphism $\varphi : [\Gamma] \to [\tilde{\Gamma}]$ between isomorphism classes of fat graphs, then the surfaces $\Sigma_{[\Gamma]}$ and $\Sigma_{[\tilde{\Gamma}]}$ are homeomorphic.

From a closed fat graph we can construct a two-dimensional cobordism. The underlying surface of the cobordism is the oriented surface Σ_Γ. This gives an orientation of the incoming and outgoing boundary component, so its enough to give a labeled marked point in each boundary component.

Note that each of the boundary components corresponds to exactly one leaf in the graph, which gives a marked point in the boundary component. We label this according to the labeling of its leaf. This gives a cobordism which is well defined up to isomorphism.

3.1.2. *Metric fat graphs.* We motivated the morphisms in the category of fat graphs thinking about lengths of edges. This is made more concrete in the space of metric fat graphs, which we describe now. This space is homeomorphic to the classifying space of the category of fat graphs, but we feel metric fat graphs are more intuitive and hence discuss them first. Several equivalent versions of this space and its dual concept (using weighted arc systems instead of fat graphs) have been studied by Harer, Penner, Igusa and Godin in [Har88, Pen87, Igu02, God04] respectively.

The idea is simple: a metric fat graph is a fat graph with lengths assigned to its edges. We need a bit more care to make this interact well with the additional data and properties of admissible fat graphs.

Definition 3.8. A *metric admissible fat graph* is a pair (Γ, λ) where Γ is an admissible fat graph and λ is a *length function*, i.e. a function $\lambda : E_\Gamma \to [0,1]$ where E_Γ is the set of edges of Γ and λ satisfies:

(i) $\lambda(e) = 1$ if e is a leaf,
(ii) $\lambda^{-1}(0)$ is a forest in Γ and $\Gamma/\lambda^{-1}(0)$ is admissible,
(iii) for any admissible cycle C in Γ we have $\sum_{e\in C} \lambda(e) = 1$.

We will call the value of λ on e the *length* of the edge e in Γ.

Definition 3.9. Suppose Γ is an admissible fat graph with p admissible cycles. Let $(n_1, n_2, \dots, n_p)$ be the number of edges on each admissible cycle and set $n := \sum_i n_i$. The *space of length functions* on Γ is given as a set by

$$\mathcal{M}(\Gamma) := \{\lambda : E_\Gamma \to [0,1] \,|\, \lambda \text{ is a length function}\}$$

There is a natural inclusion

$$\mathcal{M}(\Gamma) \hookrightarrow \Delta^{n_1-1} \times \Delta^{n_2-1} \times \dots \times \Delta^{n_p-1} \times ([0,1])^{\# E_\Gamma - n}$$

we give $\mathcal{M}(\Gamma)$ the subspace topology via this inclusion.

Definition 3.10. Two metric admissible fat graphs (Γ, λ) and $(\tilde{\Gamma}, \tilde{\lambda})$ are called *isomorphic* if there is an isomorphism of admissible fat graphs $\varphi : \Gamma \to \tilde{\Gamma}$ such that $\lambda = \tilde{\lambda} \circ \varphi_*$, where φ_* is the map induced by φ on E_Γ.

Definition 3.11. The space of *metric admissible fat graphs* is defined as

$$\mathcal{MF}at^{ad} := \frac{\bigsqcup_\Gamma \mathcal{M}(\Gamma)}{\sim}$$

where Γ runs over all admissible fat graphs and the equivalence relation $\sim$ is given by

$$(\Gamma, \lambda) \sim (\tilde{\Gamma}, \tilde{\lambda}) \iff (\Gamma/\lambda^{-1}(0), \lambda|_{E_\Gamma - \lambda^{-1}(0)}) \cong (\tilde{\Gamma}/\tilde{\lambda}^{-1}(0), \tilde{\lambda}|_{E_{\tilde{\Gamma}} - \tilde{\lambda}^{-1}(0)})$$

In other words, (i) we identify isomorphic admissible fat graphs with the same metric and (ii) we identify a metric admissible fat graph with some edges of length 0 with the metric fat graph in which these edges are collapsed and all other edge lengths remain unchanged.

Lemma 3.12. *There is a deformation retraction of the space of metric admissible fat graphs $\mathcal{MF}at^{ad}$ onto the geometric realization of the nerve of $\mathcal{F}at^{ad}$.*

Proof. We will first give a continuous map $\iota : |\mathcal{F}at^{ad}| \to \mathcal{MF}at^{ad}$ which is a homeomorphism onto its image. A point $x \in |\mathcal{F}at^{ad}|$ is represented by $x = ([\Gamma_0] \to [\Gamma_1] \to \dots \to [\Gamma_k], s_0, s_1, \dots s_k) \in N_k\mathcal{F}at^{ad} \times \Delta^k$, where N_k denotes the set of k-simplices of the nerve. Choose representatives Γ_i for $0 \le i \le k$ and for each i, let C^i_j denote the jth admissible cycle of Γ_i, n^i_j denote the number of edges in C^i_j and m^i denote the number of edges that do not belong to the admissible cycles. Each graph Γ_i naturally defines a metric admissible fat graph (Γ_0, λ_i) where λ_i is given as follows:

$$\begin{aligned} \lambda_i : E_{\Gamma_0} &\longrightarrow [0,1] \\ e &\longmapsto \begin{cases} 0 & \text{if } e \text{ is collapsed in } \Gamma_i \\ 1/n^i_j & \text{if } e \in C^i_j \\ 1/m^i & \text{otherwise} \end{cases} \end{aligned}$$

Then define $\iota(x) := (\Gamma_0, \sum_{i=0}^{k} s_i\lambda_i)$. It is easy to show that this assignment is well defined and respects the simplicial relations of the geometric realization and thus defines a continuous map. Moreover, it is injective map between Hausdorff spaces with compact image. Thus, it is a homeomorphism onto its image. We now construct a continuous map $r : \mathcal{MF}at^{ad} \times [0,1] \to \mathcal{MF}at^{ad}$ which is a strong deformation retraction of $\mathcal{MF}at^{ad}$ onto the image of ι. Since all the graphs we are considering are finite, we can define a continuous function g as follows:

$$\begin{array}{rcl} g : \mathcal{MF}at^{ad} & \longrightarrow & \mathbb{R}^{>0} \\ (\Gamma, \lambda) & \longmapsto & \sum_{e \in \tilde{E}_\Gamma} \lambda(e) \end{array}$$

where $\tilde{E}_\Gamma$ is the set of edges that do not belong to the admissible cycles. We then define r by linear interpolation as $r((\Gamma,\lambda),t) := (\Gamma, (1-t)\lambda + t\lambda_g)$, where λ_g is the rescaled length function given by:

$$\begin{array}{rcl} \lambda_g : E_\Gamma & \longrightarrow & \mathbb{R}^{\geq 0} \\ e & \longmapsto & \begin{cases} \lambda(e) & \text{if } e \text{ belongs to an admissible cycle} \\ \frac{\lambda(e)}{g(\Gamma,\lambda)} & \text{if } e \text{ does not belong to an admissible cycle} \end{cases} \end{array}$$

□

Remark 3.13. The space $\mathcal{MF}at^{ad}$ and the category $\mathcal{F}at^{ad}$ split into connected components indexed by the topological type of the graphs as two-dimensional cobordisms. That is, we have

$$\mathcal{MF}at^{ad} \cong \bigsqcup_{g,n,m} \mathcal{MF}at^{ad}_{g,n+m} \qquad\qquad \mathcal{F}at^{ad} \cong \bigsqcup_{g,n,m} \mathcal{F}at^{ad}_{g,n+m}$$

where $\mathcal{MF}at^{ad}_{g,n+m}$ and $\mathcal{F}at^{ad}_{g,n+m}$ are the connected components corresponding to admissible fat graphs with n admissible cycles which are homotopy equivalent to a surface of total genus g and $n+m$ boundary components.

3.1.3. *Sullivan diagrams.* We now define a quotient space $\mathcal{SD}$ of $\mathcal{MF}at^{ad}$, which we will see in section 5 is the analogue of the harmonic compactification for admissible fat graphs. To define it, we first describe an equivalence relation $\sim_{\mathcal{SD}}$ on metric admissible fat graphs.

Definition 3.14. We say $\Gamma_1 \sim_{\mathcal{SD}} \Gamma_2$ if Γ_2 can be obtained from Γ_1 by:

Slides: Sliding vertices along edges that do not belong to the admissible cycles.

Forgetting lengths of non-admissible edge: Changing the lengths of the edges that do not belong to the admissible cycles.

Figure 13 shows some examples of equivalent admissible fat graphs.

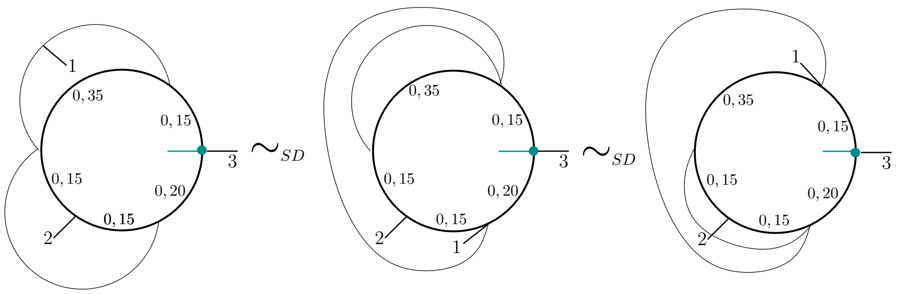

FIGURE 13. Three equivalent metric admissible fat graphs.

Definition 3.15. A *metric Sullivan diagram* is an equivalence class of metric admissible fat graphs under the relation $\sim_{\mathcal{SD}}$.

We can informally think of a Sullivan diagram as an admissible fat graph where the edges not belonging to the admissible cycles are of length zero.

Definition 3.16. The space of *Sullivan diagrams* $\mathcal{SD}$ is the quotient space $\mathcal{SD} = \mathcal{MF}at^{ad}/\!\sim_{\mathcal{SD}}$.

Remark 3.17. A path in $\mathcal{SD}$ is given by continuously moving the vertices on the admissible cycles and this space splits into connected components given by topological type.

Remark 3.18. In Section 5 we show that the space $\mathcal{SD}$ has canonical CW-complex structure. Its cellular chain complex is the complex of (cyclic) Sullivan chord diagrams introduced by Tradler and Zeinalian. It was used by them and later by Wahl and Westerland, to construct operations on the Hochschild chains of symmetric Frobenius algebras (cf. [TZ06, WW11]).

3.2. **The universal mapping class group bundle.** In this section we describe the universal mapping class group bundles over $\mathcal{F}at^{ad}$ and $\mathcal{MF}at^{ad}$. Recall that from an admissible fat graph we can construct a cobordism in which the the graph sits as a deformation retract, which depends on some choices. The idea for the construction of the universal mapping class group bundle, is to have as fiber over an admissible fat graph Γ all ways that Γ can sit in a fixed standard cobordism.

For each topological type of cobordism fix a representative surface $S_{g,n+m}$ of total genus g with n incoming boundary components and m outgoing boundary components. Fix a marked point x_k in the kth incoming boundary for $1 \leq k \leq n$ and a marked point x_{k+n} in the kth outgoing boundary $1 \leq k \leq m$.

Definition 3.19. Suppose Γ is an admissible fat graph of topological type $S_{g,n+m}$. Let $v_{\text{in},k}$ denote the kth incoming leaf and $v_{\text{out},k}$ denote the kth outgoing leaf. A *marking* of Γ is an isotopy class of embeddings $H : |\Gamma| \hookrightarrow S_{g,n+m}$ such that $H(v_{\text{in},k}) = x_k$, $H(v_{\text{out},k}) = x_{k+n}$ and the fat structure of Γ coincides with the one induced by the orientation of the surface. We will call a pair $(\Gamma, [H])$ a *marked fat graph* and we denote by $\text{Mark}(\Gamma)$ the set of *markings of* Γ.

Lemma 3.20. *Any marking $H : |\Gamma| \hookrightarrow S_{g,n+m}$ is a homotopy equivalence, and the map on π_1 induced by H sends the ith boundary cycle of Γ to the ith boundary component of $S_{g,n+m}$.*

Proof. Since the fat structure of Γ coincides with the one induced by the orientation of the surface we can thicken Γ inside $S_{g,n+m}$ to a subsurface S_Γ of the same topological type as $S_{g,n+m}$. Moreover, by the definition of a marking each boundary component of S_Γ meets a boundary component of $S_{g,n+m}$. Thus, there is a deformation retraction of $S_{g,n+m}$ onto this subsurface and onto Γ. □

Lemma 3.21. *Let Γ be an admissible fat graph, F be a forest in Γ, which does not contain any leaves of Γ. Then there is a bijection* $\text{Mark}(\Gamma) \to \text{Mark}(\Gamma/F)$ *denoted by* $[H] \mapsto [H_F]$.

This identification depends on the map connecting both graphs i.e. given $[H]$ a marking of Γ, if $\tilde{\Gamma} = \Gamma/F_1 = \Gamma/F_2$ then $[H_{F_1}]$ and $[H_{F_2}]$ can be different markings of $\tilde{\Gamma}$. Figure 14 gives an example of this in the case of the cylinder.

Proof. Let H be the representative of a marking $[H]$ of Γ. The image of $H|_F$ (the restriction of H to $|F|$) is contained in a disjoint union of disks away from the boundary. Therefore, the marking H induces a marking $H_F : |\Gamma/F| \hookrightarrow S_{g,n+m}$ given by collapsing each of the trees of F to a point of the disk in which their image is contained. Note that H_F is well defined up to isotopy and it makes the following diagram commute up to homotopy

$$\begin{tikzcd} {|\Gamma|} \arrow[r, two heads] \arrow[dr, "H"'] & {|\Gamma/F|} \arrow[d, hook, "H_F"] \\ & S_{g,n+m} \end{tikzcd}$$

In fact, up to isotopy, there is a unique embedding of a tree with a fat structure into a disk, in which the fat structure of the tree coincides with the one induced by the orientation of the disk and the endpoints are fixed points on the boundary. This can proven by induction. Start with the case where F is a single edge. Up to homotopy, there is a unique embedding of an arc in a disk where the endpoints of the arc are fixed points on the boundary. Then by [Feu66], there is also a unique embedding up to isotopy. For the induction step, let α be an arc embedded in the disk with its endpoints at the boundary and let a and b be fixed points in the boundary of a connected component of $D\backslash\alpha$. Then we have a map

$$\text{Emb}^{a,b}(I, D\backslash\alpha) \longrightarrow \text{Emb}^{a,b}(I, D)$$

where $\text{Emb}^{a,b}(I, D\backslash\alpha)$ is the space of embeddings of a path in $D\backslash\alpha$ which start at a and end at b, with the $\mathbb{C}^\infty$-topology, and similarly for $\text{Emb}^{a,b}(I, D)$. By [Gra73], this map induces injective maps in all homotopy groups, in particular in π_0, which gives the induction step.

It then follows that, given $[H_F]$ a marking of Γ/F there is a unique marking $[H]$ of Γ such that the above diagram commutes up to homotopy. $\square$

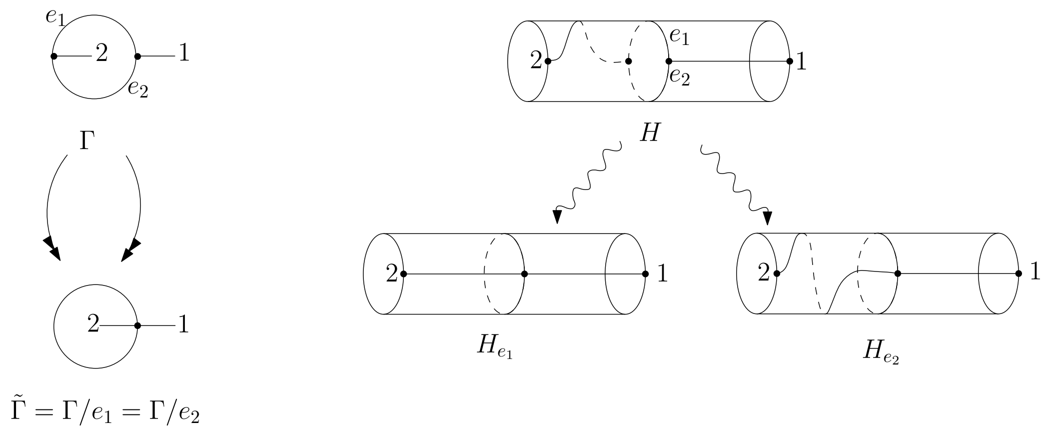

FIGURE 14. Two different embeddings of $\tilde{\Gamma}$ in the cylinder differing by a Dehn twist and corresponding to the same marking of Γ.

Definition 3.22. Define the category $\mathcal{EF}at^{ad}$ to be the category with objects isomorphism classes of marked admissible fat graphs $([\Gamma], [H])$ (where two marked admissible fat graphs are isomorphic if their underlying fat graphs are isomorphic and they have the same marking) and morphisms given by morphisms in $\mathcal{F}at^{ad}$ where the map acts on the marking as stated in the previous lemma. We denote by $\mathcal{EF}at^{ad}_{g,n+m}$, the full subcategory with objects marked admissible fat graphs whose thickening give a cobordism of topological type $S_{g,n+m}$.

Definition 3.23. The space of *marked metric admissible fat graphs* $\mathcal{EMF}at^{ad}$ is defined to be

$$\mathcal{EMF}at^{ad} := \frac{\bigsqcup_{\Gamma} \mathcal{M}(\Gamma) \times \mathrm{Mark}(\Gamma)}{\sim_E}$$

where Γ runs over all admissible fat graphs and the equivalence relation is given by

$$(\Gamma, \lambda, [H]) \sim_E (\tilde{\Gamma}, \tilde{\lambda}, [\tilde{H}]) \iff (\Gamma, \lambda) \cong (\tilde{\Gamma}, \tilde{\lambda}) \text{ and } [H_{\lambda^{-1}(0)}] = [\tilde{H}_{\tilde{\lambda}^{-1}(0)}]$$

where $\cong$ denotes isomorphism of metric fat graphs.

The following result is proven in [ES14], in fact in more generality for a category modeling open closed cobordism and not only closed cobordisms.

Theorem 3.24. *The projection* $|\mathcal{EF}at^{ad}_{g,n+m}| \to |\mathcal{F}at^{ad}_{g,n+m}|$ *is a universal* $\mathrm{Mod}(S_{g,n+m})$*-bundle.*

The proof follows the original ideas of Igusa [Igu02] and Godin [God07b]. Since all spaces involved are CW-complexes, one firstly shows that $|\mathcal{EF}at^{ad}_{g,n+m}|$ is contractible, which follows from contractibility of the arc complex [Hat91]. Secondly, one proves that the action of the mapping class group $\mathrm{Mod}(S_{g,n+m})$ on $\mathcal{EF}at^{ad}_{g,n+m}$ is free and transitive. That is, for any two markings $[H_1]$ and $[H_2]$, there is a unique $[\varphi] \in \mathrm{Mod}(S_{g,n+m})$ such that $[\varphi \circ H_1] = [H_2]$. This proof in particular gives rise to an abstract homotopy equivalence $\mathcal{M} \simeq \mathcal{F}at^{ad}$.

From Lemma 3.21 we conclude that as a set $\mathcal{EMF}at^{ad}$ is given by $\{([\Gamma, \lambda], [H]) | [\Gamma, \lambda] \in \mathcal{MF}at^{ad}, [H] \in \mathrm{Mark}([\Gamma])\}$. As before, let $\mathcal{EMF}at^{ad}_{g,n+m}$ denote the subspace of marked metric admissible fat graphs whose thickening give an open closed cobordism of topological type $S_{g,n+m}$. Then $\mathrm{Mod}(S_{g,n+m})$ acts on $\mathcal{EMF}at^{ad}_{g,n+m}$ by composition with the marking.

It follows directly that:

Corollary 3.25. *The projection* $\mathcal{EMF}at^{ad}_{g,n+m} \to \mathcal{MF}at^{ad}_{g,n+m}$ *is a universal* $\mathrm{Mod}(S_{g,n+m})$*-bundle.*

Proof. This is clear since we have a pullback diagram

$$\begin{tikzcd} \mathcal{EMF}at^{ad}_{g,n+m} \arrow[r, "r(-,1)\times \mathrm{id}", "\simeq"'] \arrow[d, two heads] & |\mathcal{EF}at^{ad}_{g,n+m}| \arrow[d, two heads] \\ \mathcal{MF}at^{ad}_{g,n+m} \arrow[r, "\simeq", "r(-,1)"'] & |\mathcal{F}at^{ad}_{g,n+m}| \end{tikzcd}$$

where the horizontal maps are the homotopy equivalences given by r, the map constructed in Lemma 3.12. □

4. The critical graph equivalence between radial slit configurations and fat graphs

4.1. **The fattening of the radial slit configurations and the critical graph map.** In Bödigheimer's construction there is a natural admissible metric fat graph associated to a configuration; the unstable critical graph. This is the graph obtained by considering inner boundaries of the annuli and the complements of the slit segments and glueing these together according to the combinatorial data. The inner boundaries of the annuli give the admissible cycles of the graph and the incoming leaves are placed at the positive real line of each annuli. The outgoing leaves are built using marked points on the outgoing boundary components. This graph gets a canonical fat graph structure from sitting inside the surface $S(L)$.

We now make this definition precise. Because we fixed the outer radii of the annuli, we shorten $\mathbb{A}^{(i)}_{R_i}$ to $\mathbb{A}_i$. To a radial slit configuration $L \in \mathfrak{QRad}$ we associate a space E_L defined as follows:

Definition 4.1. The space E_L is given by

$$E_L = \left(\bigsqcup_{1\le j\le n} \partial_{\mathrm{in}}\mathbb{A}_j\right) \sqcup \left(\bigsqcup_{1\le j\le 2h} E_j\right) \sqcup \left(\bigsqcup_{1\le j\le n} I_j\right) \sqcup \left(\bigsqcup_{1\le j\le m} E_{2h+j}\right)$$

where each of the terms are defined as follows

- **Admissible boundaries:** For each annulus $\mathbb{A}_j$ we take the inner boundary $\partial_{\mathrm{in}}\mathbb{A}_j$.
- **Leaves of the incoming cycles:** For each annulus $\mathbb{A}_j$ we have that $I_j = \{z \in \mathbb{C}_j | \arg(z) = 0, 0 \le z \le \frac{1}{2\pi}\}$.
- **Leaves of the outgoing cycles:** For $1 \le j \le m$ for each marked point $P_j \in \mathbb{A}_k$ we define $E_{2h+j} = \{z \in \mathbb{A}_k | \arg(z) = \arg(P_j)\}$.
- **Inner half edges:** For $1 \le j \le 2h$ for each slit $\zeta_j \in \mathbb{A}_k$ we define $E_j = \{z \in \mathbb{A}_k | \arg(z) = \arg(\zeta_j), |z| \le |\zeta_j|\}$.

We define an equivalence relation $\sim_L$ on E_L as the one generated by:

- **Attaching incoming leaves:** We set that $(\frac{1}{2\pi} \in I_j) \sim_L (\frac{1}{2\pi} \in \partial_{\mathrm{in}}\mathbb{A}_j)$ for $j = 1, 2, \ldots, n$.
- **Attaching inner half edges and outgoing leaves:** For $r \in \partial_{\mathrm{in}}\mathbb{A}_k$ and $e \in E_j$, we set $r \sim_L e$ if and only if $r = e$.
- **Glueing half edges:** For $e \in E_j$ and $e' \in E_k$, we set $e \sim_L e'$ if and only if $e = \zeta_j$, $e' = \zeta_k$, and $j = \lambda(k)$.
- **Identifying coinciding segments:** For $e \in E_j$ and $e' \in E_{\tilde{\omega}(j)}$, we set $e \sim_L e'$ if and only if ξ_j and $\xi_{\tilde{\omega}(j)}$ lie on the same radial segment and $|e| = |e'|$. Note that this identification only happens if $|e| = |e'| \le \min\left\{|\xi_j|, |\xi_{\tilde{\omega}(j)}|\right\}$.

Definition 4.2. For $L \in \mathfrak{QRad}$ the corresponding *critical graph* Γ_L is the underlying graph of the quotient space $E_L/{\sim_L}$ (see Figure 15).

Note that the quotient space Γ_L is invariant under the slit jump relation. Thus for a configuration $[L] \in \mathfrak{Rad}$ there is a well defined graph $\Gamma_{[L]}$. This is actually an isomorphism class of a graph since the half edges are not labeled, but we will write $\Gamma_{[L]}$ for simplicity since the choice of a representative will not affect any of the constructions below.

Furthermore, this graph is naturally embedded in the surface $\Sigma_{[L]}$ and thus it has a fat structure induced by the orientation of the surface. Moreover, this graphs is also naturally endowed with a metric $\lambda_{[L]}$ given by the standard metric in $\mathbb{C}$. This association is such that the incoming leaves

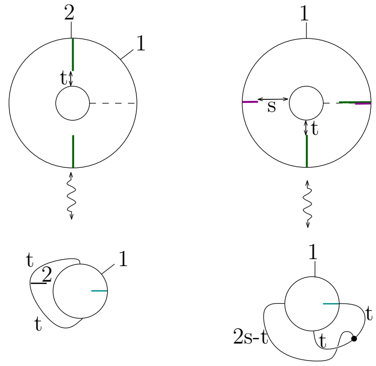

FIGURE 15. Critical graphs for different configurations.

always have fixed length and the outgoing leaves always have a strictly positive length. Because for our purposes the lengths of the outgoing leaves are superfluous information, we set $\lambda_{[L]}(e)$ to be given by the standard metric in $\mathbb{C}$ if e is not a leaf and $\lambda_{[L]}(e) = 1$ if e is a leaf. This makes $(\Gamma_{[L]}, \lambda_{[L]})$ a metric admissible fat graph.

Notation 4.3. We will just write Γ_L, when it is clear from that context that we are talking about the critical metric graph.

The construction of the critical graph gives a natural function $\mathfrak{Rad} \to \mathcal{MFat}^{ad}$ given by $[L] \mapsto (\Gamma_{[L]}, \lambda_{[L]})$. However, this function is not continuous. Discontinuities occur at non-generic configurations. To see this consider for example a path in $\mathfrak{Rad}$ given by continuously changing the the argument of a single slit as shown in Figure 16. When the moving slit reaches a neighbor slit the associated metric graph jumps.

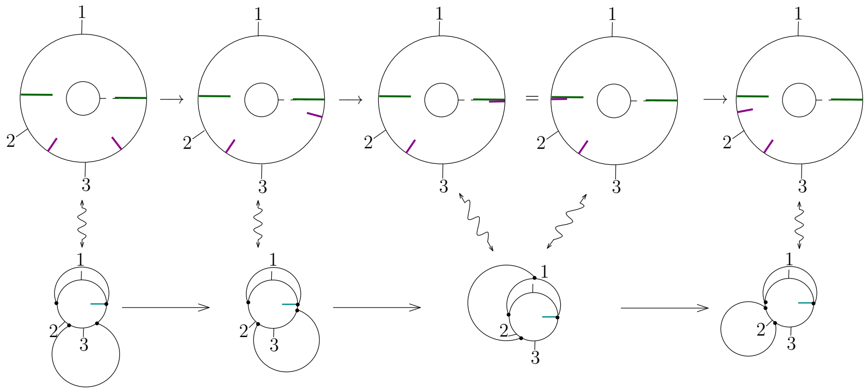

FIGURE 16. An example of a path in $\mathfrak{Rad}$ which leads to a path in $\mathcal{MFat}^{ad}$ that is not continuous.

To solve this problem we will construct a space $\mathfrak{Rad}^{\sim}$ in which we enlarge $\mathfrak{Rad}$ at non-generic configurations by a contractible space. More precisely, to define $\mathfrak{Rad}^{\sim}$ we will define a smaller equivalence relation $\sim_t$ on E_L for $L \in \mathfrak{QRad}$. This equivalence relation is supposed to only partially glue the edges in the critical graph Γ_L. To do this, we first need to introduce some notation.

Definition 4.4. Given $L \in \mathfrak{QRad}$, recall the notation $\xi_j = \zeta_j$ for $1 \leq j \leq 2h$ and $\xi_{j+2h} = P_j$ for $1 \leq j \leq m$.

(i) The ξ_i's define l distinct radial segments where $l \leq 2h + m$. These can be ordered lexicographically using the pairs (k, θ) where k is the number of the annulus to which the radial segment corresponds and θ is its argument, giving a totally ordered list of radial segments $S_1, S_2, \ldots, S_l$.

(ii) Similarly, ξ_j's that lie on S_i can be totally ordered using $\tilde{\omega}$ and $\vec{r}$. We denote by ξ_{i_j} the jth slit or parametrization point lying on S_i according to this order, i.e. the slits and parametrization

points lying on S_i are $\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_{s_i}}$ where s_i is the number of slits and marked points that lie on S_i.

Note that this notation is independent of the labeling of the slits.

Definition 4.5. Let $d_i(L) = \sum_{j=1}^{i-1}(s_j - 1)$, and $d(L) = \sum_{i=1}^{l} d_i(L)$, with s_j as given in previous definition. Suppose $t \in [0,1]^{d(L)}$, then we define an equivalence relation $\sim_t$ on the space

$$E_L = \left(\bigsqcup_{1\leq j\leq n} \partial_{\text{in}}\mathbb{A}_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq 2h} E_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq n} I_j\right) \sqcup \left(\bigsqcup_{1\leq j\leq m} E_{j+2h}\right)$$

to be the one generated by:

Attaching inner leaves: We set $(\frac{1}{2\pi} \in I_j) \sim (\frac{1}{2\pi} \in \partial_{\text{in}}\mathbb{A}_j)$ for $j = 1, 2, \ldots, n$.

Attaching inner half edges and outgoing leaves: For $r \in \partial_{\text{in}}\mathbb{A}_j$ and $e \in E_j$ we set $r \sim_t e$ if and only if $r = e$.

Glueing half edges: For $1 \leq j, k \leq 2h$, $e \in E_j$ and $e' \in E_k$ we set $e \sim_t e'$ if and only if $e = \zeta_j$, $e' = \zeta_k$, and $j = \lambda(k)$.

Partially identifying coinciding segments: For $1 \leq i \leq l$, $1 \leq j \leq s_i - 1$, $e \in E_{i_j}$ and $e' \in E_{i_{j+1}}$ we set $e \sim_t e'$ if and only if

$$|e| = |e'| \leq t_{d_{i_j}} + \frac{1}{2\pi}$$

Notice that the conditions imply that ξ_{i_j} and $\xi_{i_{j+1}}$ lie on the same radial segment, namely S_i. In words, we identify the edges corresponding to these slits or parametrization points up to distance $t_{d_{i_j}}$ from the admissible boundary.

Definition 4.6. We define $\Gamma_{L,t}$ to be the underlying graph of the quotient space $E_L/\sim_t$. If $t = (0, 0, 0, \ldots, 0)$ we will call this the *unfolded graph of* L and denote it $\Gamma_{L,0}$ (see Figure 17).

Notice that as before $\Gamma_{L,t} \in \mathcal{F}at^{ad}$ since it has a naturally associated fat structure. For $t = (1, 1, 1, \ldots, 1)$ we have that $\Gamma_{L,t}$ is the critical graph Γ_L, which is invariant under slit and parametrization points jumps. However, for most other t, the graph $\Gamma_{L,t}$ is not invariant under slit jumps, so it is not well defined for $[L] \in \mathfrak{Rad}$. Just like the critical graph, the graph $\Gamma_{L,t}$ has a natural metric making $(\Gamma_{L,t}, \lambda_{L,t})$ an admissible metric fat graph. Figure 17 shows examples of unfolded and partially unfolded metric admissible fat graphs of a specific configuration. Two preconfigurations with the same combinatorial type have the same (non-metric) admissible fat graphs but with different length functions. Thus it makes sense to talk about $\Gamma_{\mathcal{L},t}$ which is a (non-metric) admissible fat graph. Similarly, it makes sense to talk about the critical graph of a combinatorial type, which we denote $\Gamma_{[\mathcal{L}]}$.

Definition 4.7. Let $[L] \in \mathfrak{Rad}$, we define a subspace of $\mathcal{MF}at^{ad}$

$$\mathcal{G}([L]) := \{[\Gamma_{L_i,t}, \lambda_{L_i,t}] | [L] = [L_i], t \in [0,1]^{d(L_i)}\}.$$

We define the *fattening of* $\mathfrak{Rad}$ to be the space

$$\mathfrak{Rad}^{\sim} = \{([L], [\Gamma, \lambda]) \in \mathfrak{Rad} \times \mathcal{MF}at^{ad} | [\Gamma, \lambda] \in \mathcal{G}([L])\}.$$

For simplicity, we will just write $\Gamma_{L_i,t}$ or Γ when it is clear from the context that we are talking about metric graphs.

We will show that $\mathfrak{Rad}^{\sim}$ is constructed by replacing the point $[L] \in \mathfrak{Rad}$ by a contractible space, $\mathcal{G}([L])$, which is a family of graphs that "interpolates" between the critical graph of $[L]$ and the unfolded graphs of the different representatives $L_1, L_2, \ldots, L_k$ of $[L]$ in $\mathfrak{QRad}$.

Proposition 4.8. *The space $\mathcal{G}([L])$ is a contractible space.*

Proof. We define a homotopy h_s from $\mathcal{G}([L])$ onto the point Γ_L as follows. Fix a representative L of $[L]$. Recall from Definitions 4.4 and 4.5 that the coordinates of $[0,1]^{d(L)}$ are indexed first by radial segments S_i and then by consecutive slits $(\xi_{i_j}, \xi_{i_{j+1}})$ bordering these segments. Let $\sim_{h_s(t)}$ be the equivalence relation on E_L as in Definition 4.5 except that the equation for partially identifying coinciding segments is replaced by: $e \sim_t e'$ if and only if

$$|e| = |e'| \leq \max\{t_{d_{i_j}}, s\} + \frac{1}{2\pi}$$

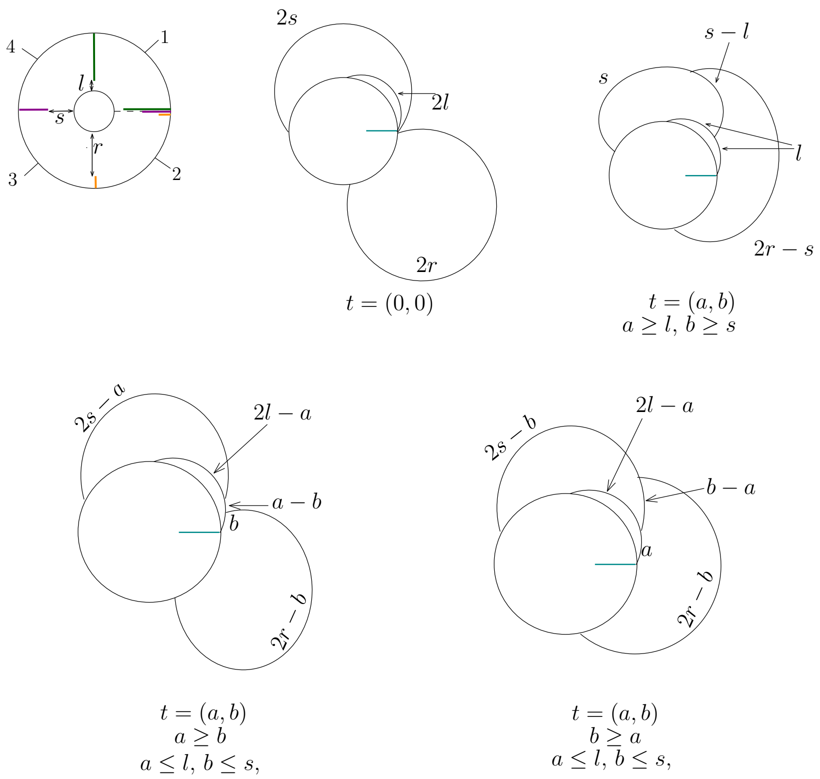

FIGURE 17. A configuration $[L]$ on the top left, and several graphs obtained from it using different t. The leaves have been omitted from the unfolded graphs to make them more readable, but in this case they are all located along the admissible cycles according to the positions of the marked points in $[L]$.

We then set the homotopy to be

$$h_s(\Gamma_{L_i,t}) = E_L/\sim_{h_s(t)}$$

which can be described in words as increasingly glueing edges together as one moves outwards from the admissible cycles at constant speed.

We need to check that this is well-defined and continuous. The continuity follows directly from the continuous dependence of the edge lengths on s. To see it is independent of representative $\Gamma_{L,t}$ for Γ, suppose that $\Gamma_{L,t} = \Gamma_{L',t'}$. Then L differs from L' via some slits or parametrization point jumps. The corresponding coordinates for those slits or parametrization points must either be equal to each other, or both be greater or equal to than $\min\{|\zeta_{i_j}|, |\zeta_{i_{j+1}}|\}$. In the former case h_s leaves both coordinates unchanged and in the latter cases replaces both by their maximum with s, preserving equality. □

The blow up of $\mathfrak{Rad}$ splits into connected components given by the topological type of the cobordism they describe. That is

$$\mathfrak{Rad}^{\sim} := \bigsqcup_{h,n,m} \mathfrak{Rad}_h^{\sim}(n, m)$$

Moreover, this space comes with two natural maps

$$\mathfrak{Rad} \xleftarrow{\pi_1} \mathfrak{Rad}^{\sim} \xrightarrow{\pi_2} \mathcal{MF}at^{ad}$$

We call π_1 the *projection map* and π_2 the *critical graph map*. In the remaining subsections we will show that these maps are homotopy equivalences.

4.2. **The projection map is a homotopy equivalence.** We want to prove that the projection map $\pi_1 : \mathfrak{Rad}^{\sim} \to \mathfrak{Rad}$ is a homotopy equivalence. The idea of the proof is as follows: both $\mathfrak{Rad}^{\sim}$ and $\mathfrak{Rad}$ are nice spaces and π_1 is a nice map with contractible fibers, so it is a homotopy equivalence.

To make this statement precise, we need to replace "nice" with actual mathematical content. The precise statement is as follows: $\mathfrak{Rad}$ and $\mathfrak{Rad}^\sim$ are *absolute neighborhood retracts* (henceforth ANR's), and $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is a proper cell-like map. We can then use the following result of Lacher, more precisely the Theorem on page 510 of [Lac77].

Theorem 4.9 (Lacher)**.** *A proper map $f : X \to Y$ between locally compact ANR's is cell-like if and only if for all opens $U \subset Y$ the restriction $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a proper homotopy equivalence.*

We will now define the terms that appear in this theorem.

Definition 4.10. A subspace X of a space Y is a *neighborhood retract* if there exists an open subset U of Y containing X and a retraction $r : U \to X$. A space X is an *ANR* if it has the property that if X is a closed subspace of a metric space Y, then X is a neighborhood retract of Y.

Definition 4.11.
(i) A subset A of a manifold M is *cellular* if it is the intersection $\bigcap_n E_n$ of a nested countable sequence $E_1 \supset E_2 \supset \ldots$ of n-cells E_i in a manifold M, i.e. subsets homeomorphic to D^n.
(ii) A space X is *cell-like* if there is an embedding (i.e. continuous map that is an homomorphism onto its image) $\phi : X \to M$ of X into a manifold, such that $\phi(X)$ is cellular.
(iii) A map $f : X \to Y$ is *cell-like* if for all $y \in Y$ the point inverse $f^{-1}(\{y\})$ is a cell-like space.

Both of these definitions are sufficiently abstract that it is hard to apply them directly. Our main reference for ANR's is [vM89], for polyhedra is Chapter 3 of [FP90] and for cell-like spaces is [Lac77]. We will now give a proposition stating the properties of ANR's and cell-like spaces.

Proposition 4.12. *The following are properties of ANR's:*
(i) For all $n \geq 0$, D^n is an ANR.
(ii) An open subset of an ANR is an ANR.
(iii) If X is a space with an open cover by ANR's, then X is an ANR.
(iv) If X and Y are compact ANR's, $A \subset X$ is a compact ANR and $f : A \to Y$ is continuous, then $X \cup_f Y$ is an ANR.
(v) Any locally finite CW-complex is an ANR.
(vi) Any locally finite polyhedron is an ANR.
(vii) A product of finitely many ANR's is an ANR.
(viii) A compact ANR is cell-like if and only if it is contractible.

Proof. Property (i) follows from Corollary 5.4.6 of [vM89], property (ii) is Theorem 5.4.1, property (iii) is theorem 5.4.5, property (iv) is Theorem 5.6.1. Together these can combined to prove property (v), by noting that by (ii) and (iii) one can reduce to the case of finite CW-complex and since by definition these can be obtained by glueing closed n-disks together, (i) and (iv) prove that finite CW-complexes are ANR's. Property (vi) follows from property (v), but is also Theorem 3.6.11 of [vM89]. Property (vii) is Proposition 1.5.7. Finally, property (viii) follows from Theorem 4.9 by considering the map to a point. $\square$

Our next goal is to check that the spaces $\mathfrak{Rad}$ and $\mathfrak{Rad}^\sim$ are ANR's and that the map $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is proper and cell-like.

Proposition 4.13. *The space $\mathfrak{Rad}$ is an ANR.*

Proof. The space $\mathfrak{Rad}$ is a smooth manifold and has a cover by $\mathbb{R}^n$'s, and one can use properties (i), (ii) and (iii). These are ANR's by property (v) of Proposition 4.12. Alternatively one can argue that $\mathfrak{Rad}$ is an open subspace of the finite CW-complex $\overline{\mathfrak{Rad}}$ and use properties (ii) and (v) of Proposition 4.12. $\square$

To prove that $\mathfrak{Rad}^\sim$ is an ANR and that π_1 is a proper cell-like map, we will write $\mathfrak{Rad}^\sim$ as an open subspace of a space $(\overline{\mathfrak{Rad}})^\sim$ obtained by glueing together finitely many compact ANR's. This requires us to first check that $\mathcal{G}([L])$ is an ANR, and we will in fact prove a bit more.

Lemma 4.14. *For all $[L]$ the space $\mathcal{G}([L])$ is a compact polyhedron and thus a compact ANR.*

Proof. The space $\mathcal{G}([L])$ is a subspace of $\mathcal{MFat}^{ad}_{g,n+m}$. The latter is contained in the larger compact polyhedron given by

$$P_{g,n+m} := \frac{\bigsqcup_\Gamma \Delta^{n_1-1} \times \Delta^{n_2-1} \times \cdots \times \Delta^{n_p-1} \times ([0,1])^{\#E_\Gamma - n}}{\sim}$$

with Γ indexed by the objects of $\mathcal{Fat}^{ad}_{g,n+m}$ and the equivalence relation $\sim$ given by Definition 3.7. This is compact because $\mathcal{Fat}^{ad}_{g,n+m}$ has finitely many objects.

The subspace $\mathcal{G}([L])$ can be characterized as the union of the images of the maps $[0,1]^{d(L_i)} \to \mathcal{MFat}^{ad}_{g,n+m}$ for all representatives L_i of $[L]$. Each of these map is a piecewise linear map between polyhedra, which implies that their image is a subpolyhedron. This is true because a piecewise linear map by definition can be made simplicial with respect to some triangulation and the images of simplicial maps are clearly polyhedra. Note that there are only finitely many representatives for $[L]$, so that $\mathcal{G}([L])$ is a union of finitely many compact polyhedra, which implies it is a polyhedron by Corollary 3.1.27 of [FP90]. The last claim then follows from property (vi) of Proposition 4.12. ☐

Note that by Definition 2.17 we have $\overline{\mathfrak{Rad}} \backslash \mathfrak{Rad} = \mathfrak{Rad}'$ is a CW-complex, and in fact a subcomplex of $\overline{\mathfrak{Rad}}$. Then $(\overline{\mathfrak{Rad}})^\sim$ is defined by simply adding a boundary to the blowup $\mathfrak{Rad}^\sim$ in the most naive way.

Definition 4.15. Fix g, n and m. The space $(\overline{\mathfrak{Rad}})^\sim$ is the subspace of $\overline{\mathfrak{Rad}} \times P$ consisting of all pairs $([L], \Gamma, \lambda)$ such that either (i) $[L] \in \mathfrak{Rad}$ and $(\Gamma, \lambda) \in \mathcal{G}(L)$, or (ii) $[L] \in \overline{\mathfrak{Rad}} \backslash \mathfrak{Rad}$ and $(\Gamma, \lambda) \in P$.

Lemma 4.16. *Fix g, n and m, then the space $(\overline{\mathfrak{Rad}})^\sim$ is a compact ANR.*

Proof. Fix a representative $[L]$ for each combinatorial type $[\mathcal{L}]$ and note that if $[L]$ and $[L']$ have the same combinatorial type, there is a canonical homeomorphism $\mathcal{G}([L]) \cong \mathcal{G}([L'])$. The space $\mathcal{G}([\mathcal{L}])$ is then by definition $\mathcal{G}([L])$ for the representative $[L]$ of $[\mathcal{L}]$. Remark that $(\overline{\mathfrak{Rad}})^\sim$ is obtained by glueing together $\overline{\mathfrak{Rad}} \backslash \mathfrak{Rad} \times P$ and $\overline{\mathfrak{Rad}}_{[\mathcal{L}]} \times \mathcal{G}([\mathcal{L}])$ for all combinatorial types $[\mathcal{L}]$ along $\partial \overline{\mathfrak{Rad}}_{[\mathcal{L}]} \times \mathcal{G}([\mathcal{L}])$.

Note that $\overline{\mathfrak{Rad}} \backslash \mathfrak{Rad} \times P$ is the product of a subcomplex of the finite complex $\overline{\mathfrak{Rad}}$ with a compact polyhedron. Thus parts (v) and (vii) of Proposition 4.12 say it is a compact ANR. Similarly, by Lemma 4.14 we have that $\overline{\mathfrak{Rad}}_{[\mathcal{L}]} \times \mathcal{G}([\mathcal{L}])$ and $\partial \overline{\mathfrak{Rad}}_{[\mathcal{L}]} \times \mathcal{G}([\mathcal{L}])$ are each a product of a finite CW-complex with a compact polyhedron, and thus compact ANR's by parts (v), (vi) and (vii) of Proposition 4.12. Attaching cells $\overline{\mathfrak{Rad}}_{[\mathcal{L}]}$ one at a time in order of dimension and repeatedly applying property (iv) of Proposition 4.12, one proves inductively over k that

$$\left(\overline{\mathfrak{Rad}} \backslash \mathfrak{Rad} \times P\right) \cup \left(\bigcup_{\dim \overline{\mathfrak{Rad}}_{[\mathcal{L}]} \leq k} \overline{\mathfrak{Rad}}_{[\mathcal{L}]} \times \mathcal{G}([\mathcal{L}]) \right)$$

is a compact ANR. This uses that $\overline{\mathfrak{Rad}}$ only has finitely many cells after fixing g, n and m. In particular this process has to end at some $k \geq 0$ and we conclude that $(\overline{\mathfrak{Rad}})^\sim$ is also a compact ANR. ☐

Proposition 4.17. *Fix g, n and m, then the space $\mathfrak{Rad}^\sim$ is an ANR.*

Proof. $\mathfrak{Rad}^\sim$ is an open subspace of $(\overline{\mathfrak{Rad}})^\sim$ and by property (ii) of Proposition 4.12 we conclude it is an ANR. ☐

Proposition 4.18. *The map $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is proper and cell-like.*

Proof. We note that π_1 extends to a continuous map $\bar{\pi}_1 : (\overline{\mathfrak{Rad}})^\sim \to \overline{\mathfrak{Rad}}$. Let $K \subset \mathfrak{Rad}$ be compact, then it is also compact considered as a subset of $\overline{\mathfrak{Rad}}$ and thus closed. This means that $\bar{\pi}_1^{-1}(K)$ is closed in $(\overline{\mathfrak{Rad}})^\sim$ and since the latter is a compact space it must compact. But $\bar{\pi}_1^{-1}(K) \subset \mathfrak{Rad}^\sim$ and $\bar{\pi}_1^{-1}(K) \cap \mathfrak{Rad}^\sim = \pi_1^{-1}(K)$, so that π_1 is proper.

That π_1 is cell-like is a consequence of Lemma's 4.8 and 4.14, which say that the point inverses of π_1 are contractible compact polyhedra, and property (viii) in Proposition 4.12, which implies that contractible compact polyhedra are cell-like. ☐

Corollary 4.19. *The projection $\pi_1 : \mathfrak{Rad}^\sim \to \mathfrak{Rad}$ is a homotopy equivalence.*

Proof. It suffices to prove this for fixed g, n and m. Then we can simply apply Theorem 4.9 to Propositions 4.13, 4.17 and 4.18. ☐

4.3. **The critical graph map is a homotopy equivalence.** We now show that the critical graph map $\mathfrak{Rad}^{\sim} \to \mathcal{MFat}^{ad}$ is a homotopy equivalence using the relation between the universal bundles over $\mathfrak{Rad}$ and $\mathcal{MFat}^{ad}$. We start by recalling some well-known results regarding universal bundles.

Proposition 4.20. *Given a two-dimensional cobordism $S_{g,n+m}$ and a paracompact base space B, there are bijections between*

(i) isomorphism classes of smooth $S_{g,n+m}$-bundles over B, i.e. the transition functions lie in $\mathrm{Diff}(S_{g,n+m})$*,*
(ii) isomorphism classes of principal $\mathrm{Diff}(S_{g,n+m})$*-bundles over B, and*
(iii) isomorphism classes of principal $\mathrm{Mod}(S_{g,n+m})$*-bundles over B.*

The bijections are natural in B.

$$\begin{array}{c}\{\text{isomorphism classes of smooth } S_{g,n+m}\text{-bundles over } B\}\\ \updownarrow\wr \\ \{\text{isomorphism classes of principal } \mathrm{Diff}(S_{g,n+m})\text{-bundles over } B\}\\ \updownarrow\wr \\ \{\text{isomorphism classes of principal } \mathrm{Mod}(S_{g,n+m})\text{-bundles over } B\}\end{array}$$

Sketch of proof. For one direction of the first bijection, consider a principal $\mathrm{Diff}(S_{g,n+m})$-bundle $p : W \to B$. Its corresponding $S_{g,n+m}$-bundle is given by taking $S_{g,n+m} \times_{\mathrm{Diff}(S_{g,n+m})} W$.

For the other direction of the first bijection, suppose that $\pi : E \to B$ is a smooth $S_{g,n+m}$-bundle. Each fiber $E_b := \pi^{-1}(b)$ is a Riemann surface with boundary, together with a marked point in each boundary component. These marked points are ordered and labeled as incoming or outgoing. Let x_k^b denote the marked point in the kth incoming boundary component for $1 \leq k \leq n$ and x_{k+n}^b denote the marked point in the kth outgoing boundary $1 \leq k \leq m$. Its corresponding $\mathrm{Diff}(S_{g,n+m}, \partial S_{g,n+m})$-bundle is given by taking fiberwise orientation-preserving diffeomorphisms i.e. it is the bundle $p : W \to B$ whose fibers are given by

$$W_b := p^{-1}(b) = \{\varphi : S_{g,n+m} \to E_b \,|\, \varphi \text{ is a diffeomorphism, } \varphi(x_i) = x_i^b\}$$

These constructions are mutually inverse.

Furthermore, each connected component of $\mathrm{Diff}(S_{g,n+m})$ is contractible, so taking π_0 is a homotopy equivalence and thus there is a one-to-one correspondence between principal $\mathrm{Diff}(S_{g,n+m})$-bundles and principal $\mathrm{Mod}(S_{g,n+m})$-bundles, where one can obtain the $\mathrm{Mod}(S_{g,n+m})$-bundle corresponding to $p : W \to B$ by taking π_0. □

We now construct a space $E\mathfrak{Rad}$ that maps onto $\mathfrak{Rad}$ and use the relations above to show that $E\mathfrak{Rad} \to \mathfrak{Rad}$ is a universal $\mathrm{Mod}(S_{g,n+m})$-bundle. To construct this space we use the same ideas as used in the construction of $\mathcal{EMFat}^{ad}$ in Definition 3.23. That is, as a set we define

$$E\mathfrak{Rad} := \{([L],[H]) \,|\, [L] \in \mathfrak{Rad}, [H] \text{ is a marking of } \Gamma_{[L]}\}$$

We give a topology on $E\mathfrak{Rad}$ such that the map $E\mathfrak{Rad} \to \mathfrak{Rad}$ is a covering map. Then a path in $E\mathfrak{Rad}$ will be given by a path $\gamma : t \to [L(t)]$ in $\mathfrak{Rad}$ together with a marking $H_0 : \Gamma_{[L(0)]} \hookrightarrow S_{g,n+m}$. Hence we must describe how H_0 and the path γ uniquely determine a sequence of markings $H_t : \Gamma_{[L(t)]} \hookrightarrow S_{g,n+m}$. To make this precise, we will give a procedure to obtain a well defined marking of $\Gamma_{[\tilde{\mathcal{L}}]}$ from a combinatorial type $[\mathcal{L}]$, a marking of $\Gamma_{[\mathcal{L}]}$ and a configuration $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}}_{[\mathcal{L}]}$, where $[\tilde{\mathcal{L}}]$ is the combinatorial type of $[\tilde{L}]$. To describe this procedure, notice that if $[\mathcal{L}]$ and $[\tilde{\mathcal{L}}]$ are related in this manner, then $[\tilde{\mathcal{L}}]$ must be obtained from $[\mathcal{L}]$ by collapsing radial and annular chambers. Hence, we will start by analyzing these cases separately. We start by defining an annular chamber collapse map.

Definition 4.21. Let $[\mathcal{L}]$ and $[\mathcal{L}']$ be two non degenerate combinatorial types such that $[\mathcal{L}']$ can be obtained from $[\mathcal{L}]$ by collapsing the annular chambers $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ and let $A := \cup_i A_i$. We

will define a map in $\mathcal{F}at^{ad}$

$$\rho : \Gamma_{[\mathcal{L}]} \to \Gamma_{[\mathcal{L}']}$$

which we will call the *annular chamber collapse map* (see Figure 18).

Choose a representative $[L]$ of $[\mathcal{L}]$. Then following the construction of $\Gamma_{[L]}$ we can define a subgraph F_A which is given by the intersection of E_L and A. The subgraph F_A must be a forest inside $\Gamma_{[L]}$. To see this, assume there is a loop in F_A, then there must be a loop in $\Gamma_{[L]}$, this means that there are two paired slits ζ_i, $\zeta_{\lambda(i)}$ which lie on the same radial segment. Since $[L]$ is non degenerate the there must be slits $\zeta_{i_1}, \zeta_{i_2}, \ldots, \zeta_{i_j}$ such that $i_j \geq 1$ and $|\zeta_{i_l}| < |\zeta_i|$ for all i_l. Finally, since the loop is in F_A, then A must contain the radial segment between ζ_i and ζ_{i_l} for some i_l, but then collapsing A will give a degenerate configuration and we assumed $[\mathcal{L}']$ is non degenerate. Therefore F_A is a forest in $\Gamma_{[L]}$ and since $\Gamma_{[L]} = \Gamma_{[\mathcal{L}]}$ this description gives a well defined subforest of $\Gamma_{[\mathcal{L}]}$ giving with a well defined map on $\mathcal{F}at^{ad}$.

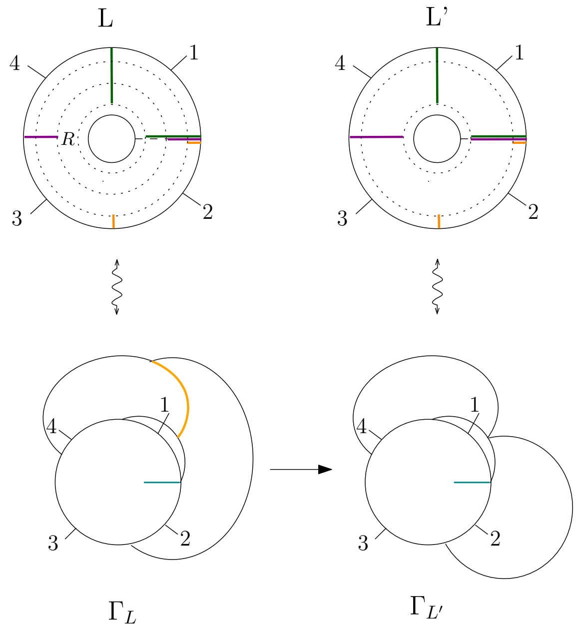

FIGURE 18. An example of the annular chamber collapse map. The annular chambers are marked with dotted lines. The radial sector R is collapsed in L and the annular chamber collapse map is given by contracting the edge shown in orange.

Next we define a radial chamber collapse zigzag.

Definition 4.22. Let $[\mathcal{L}]$ and $[\mathcal{L}'']$ be two non degenerate combinatorial types such that $[\mathcal{L}'']$ can be obtained from $[\mathcal{L}]$ by collapsing radial chambers. We will define an admissible fat graph $\Gamma([\mathcal{L}], [\mathcal{L}''])$ together with a zigzag in $\mathcal{F}at^{ad}$

$$\Gamma_{[\mathcal{L}]} \xrightarrow{\tau_1} \Gamma([\mathcal{L}], [\mathcal{L}'']) \xleftarrow{\tau_2} \Gamma_{[\mathcal{L}']}$$

which we will call the *radial chamber collapse zigzag* (see Figure 19).

Choose a representative $L \in \mathfrak{QRad}$ of combinatorial type $[\mathcal{L}]$ and let $L'' \in \mathfrak{QRad}$ be the preconfiguration of combinatorial type $[\mathcal{L}'']$ obtained by collapsing radial chambers. We will call the radial segments onto which the radial chambers have been collapsed the *special radial segments.* Notice that L'' is well defined up to a choice of L, and slit jumps and parametrization point jumps away from the special radial segments. Thus the idea is to define $\Gamma([\mathcal{L}], [\mathcal{L}''])$ as a partially unfolded graph of L'' which is unfolded at the special radial slit segments and folded everywhere else. This would give a well defined isomorphism class of admissible fat graphs.

To make this precise, let $S_{k_1}, S_{k_2}, \ldots, S_{k_r}$ denote the special radial segments of L''. We define $\Gamma([\mathcal{L}], [\mathcal{L}'']) = \Gamma_{L'',t}$ where $t \in [0,1]^{d(L'')}$ is defined as follows:

$$t_\alpha := \begin{cases} 0 & \text{if } \alpha = k_i + j \text{ for } 1 \leq i \leq r \text{ and } 1 \leq j \leq s_{k_i} - 1 \\ 1 & \text{else} \end{cases}$$

This is a well defined isomorphism class of admissible fat graphs, since the graph is folded in all radial segments in which jumps are allowed. Let F_L be the subgraph of Γ_L obtained by the intersection of E_L with the collapsing chambers. Then $\tau_1 : \Gamma_{[\mathcal{L}]} = \Gamma_L \to \Gamma_L/F_L = \Gamma([\mathcal{L}], [\mathcal{L}''])$ is a well defined map in $\mathcal{F}at^{ad}$. Similarly let $F_{L''}$ be the subgraph of $\Gamma_{L''}$ obtained from the intersection of $E_{L''}$ and the special radial segments. Then $\tau_2 : \Gamma_{[\mathcal{L}'']} = \Gamma_{L''} \to \Gamma_{L''}/F_{L''} = \Gamma([\mathcal{L}], [\mathcal{L}''])$ is a well defined map in $\mathcal{F}at^{ad}$.

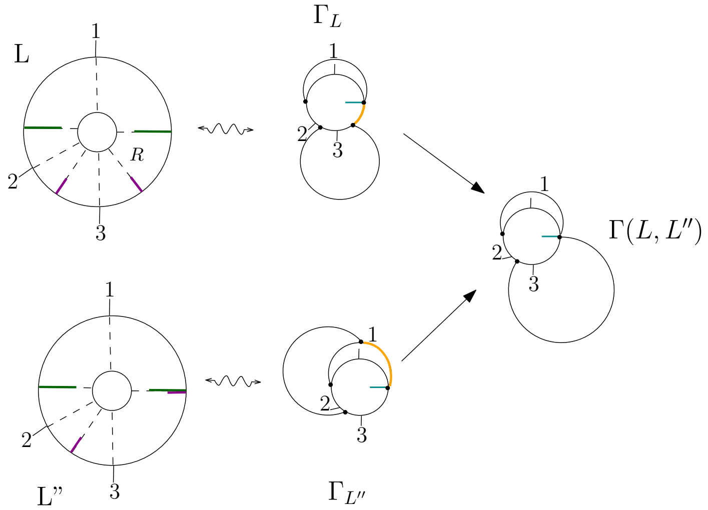

FIGURE 19. An example of the radial chamber collapse zigzag. The radial chambers are marked with dotted lines. The radial chamber R is collapsed in L and the radial chamber collapse zigzag is given by collapsing the edges shown in orange.

For the general case consider any $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}_{[\mathcal{L}]}} \cap \mathfrak{Rad}_{[\tilde{\mathcal{L}}]}$. Then $[\tilde{\mathcal{L}}]$ is obtained from $[\mathcal{L}]$ by collapsing chambers. If we let $[\mathcal{L}']$ be the configuration obtained from collapsing only the annular chambers, then the previous construction gives a well-defined zigzag in $\mathcal{F}at^{ad}$.

$$\Gamma_{[\mathcal{L}]} \xrightarrow{\rho} \Gamma_{[\mathcal{L}']} \xrightarrow{\tau_1} \Gamma([\mathcal{L}'], [\mathcal{L}]) \xleftarrow{\tau_2} \Gamma_{[\mathcal{L}']} \tag{1}$$

Note that if $[\tilde{\mathcal{L}}]$ is obtained by only collapsing annular chambers then $\tau_1 = \mathrm{id} = \tau_2$ and if $[\tilde{\mathcal{L}}]$ is obtained by only collapsing radial chambers then $\rho = \mathrm{id}$.

Definition 4.23. We define the space $E\mathfrak{Rad}$ as follows

$$E\mathfrak{Rad} := \frac{\bigsqcup_{[\mathcal{L}]} \mathfrak{Rad}_{[\mathcal{L}]} \times \mathrm{Mark}(\Gamma_{[\mathcal{L}]})}{\sim}$$

where the disjoint union runs over all non degenerate combinatorial types $[\mathcal{L}]$ and the equivalence relation $\sim$ is generated by saying that $([\tilde{L}], [H]) \sim ([\tilde{L}], [\tilde{H}])$ if given $[\tilde{L}] \in \partial\overline{\mathfrak{Rad}_{[\mathcal{L}]}} \cap \mathfrak{Rad}_{[\tilde{\mathcal{L}}]}$, $[H] \in \mathrm{Mark}(\Gamma_{[\mathcal{L}]})$, $[\tilde{H}] \in \mathrm{Mark}(\Gamma_{[\tilde{\mathcal{L}}]})$ we have that $[\tilde{H}] = (\tau_{2*})^{-1} \circ (\tau_{1*}) \circ \rho_*([H])$. Here ρ, τ_1 and τ_2 are given as in diagram 1 and the induced maps are the ones constructed in Remark 3.21.

Proposition 4.24. *The projection $E\mathfrak{Rad} \to \mathfrak{Rad}$ is a universal $\mathrm{Mod}(S_{g,n+m})$-bundle over $\mathfrak{Rad}$.*

Proof. It is enough to show that $E\mathfrak{Rad} \to \mathfrak{Rad}$ is the $\mathrm{Mod}(S_{g,n+m})$-bundle corresponding to the universal surface bundle $p : \mathrm{S}_h(n,m) \to \mathrm{Rad} \cong \mathfrak{Rad}$. Recall that the universal surface bundle has fibers $p_{[L]} = S([L])$, a surface with boundary with a marked point in each boundary component. These marked points are ordered and labeled as incoming or outgoing.

Let x_k^L denote the marked point in the kth incoming boundary component for $1 \leq k \leq n$ and x_{k+n}^L denote the marked point in the kth outgoing boundary $1 \leq k \leq m$. Following the description in the beginning of this subsection, the $\mathrm{Diff}(S_{g,n+m})$-bundle $W \to \mathfrak{Rad}$, corresponding

to the universal surface bundle is given by taking fiberwise orientation preserving diffeomorphisms. That is, we have

$$W_{[L]} := \{\varphi : S_{g,n+m} \to S([L]) \,|\, \varphi \text{ is an orientation-preserving diffeomorphism with } \varphi(x_i) = x_i^L\}$$

Furthermore, its corresponding $\mathrm{Mod}(S_{g,n+m})$-bundle $Q \to \mathfrak{Rad}$, has fibers $Q_{[L]} := W_{[L]}/\text{isotopy}$. This amounts to passing to connected components of the group of diffeomorphisms.

Note that $Q_{[L]}$ is discrete, and thus by the description of $E\mathfrak{Rad}$ it is enough to show that there is a bijection between $\mathrm{Mark}(\Gamma_{[L]})$ and $Q_{[L]}$. We define inverse maps

$$\Phi : Q_{[L]} \leftrightarrows \mathrm{Mark}(\Gamma_{[L]}) : \Psi$$

By construction, there is a canonical embedding $H_{[L]} : \Gamma_{[L]} \hookrightarrow S([L])$ and this embedding is a marking of $\Gamma_{[L]}$ in $S([L])$. Given $[\varphi] \in Q_{[L]}$ we define $\Phi([\varphi]) := [\varphi^{-1} \circ H_{[L]}]$, this is a well defined map.

To go back, let $[H] \in \mathrm{Mark}(\Gamma_{[L]})$ and choose a representative $H : \Gamma_{[L]} \hookrightarrow S_{g,n+m}$. We will construct an orientation preserving homeomorphism $f : S_{g,n+m} \to S([L])$ such that $[f \circ H] = [H_{[L]}]$, which we can approximate by a diffeomorphism φ, by Nielsen's approximation theorem [Nie24]. To do so, we use that the complements of the markings are disks and construct the homeomorphism by first on markings and then extending them to disks.

By 3.20, the complement $S_{g,n+m} \backslash H(\Gamma \backslash \text{leaves of } \Gamma)$ is a disjoint union of $n+m$ cylinders. For all $1 \leq i \leq n+m$, one of the boundary components of the ith cylinder consists of the ith boundary of $S_{g,n+m}$. The other boundary component consists of the image of the ith boundary cycles of Γ under H. The leaf corresponding to the ith boundary component is embedded in the cylinder and connects both boundary components. We conclude that $S_{g,n+m} \backslash H(\Gamma_{[L]}) \cong \bigsqcup_{i=1}^{n+m} D_i$ where each D_i is a disk.

Let x_i denote the marked point of the ith boundary component of $S_{g,n+m}$. The boundary of D_i has two copies of x_i. Connecting these on one side is the ith boundary component of $S_{g,n+m}$ and on the other side the embedded image of the ith boundary cycle of $\Gamma_{[L]}$. The orientation of the ith boundary component of $S_{g,n+m}$ allows us to order the two copies of x_i and label them as $x_{i,1}$ and $x_{i,2}$ respectively. Similarly, we have that $S([L]) \backslash H_{[L]}(\Gamma_{[L]}) \cong \bigsqcup_{i=1}^{n+m} \tilde{D}_i$ where each $\tilde{D}_i$ is a disk.

Let $x_{i,j}^L$ for $j = 1,2$ denote the two copies of the marked point on the ith boundary component of $S([L])$, that lie on the boundary of $\tilde{D}_i$. Take $f_i|_{\partial D_i} : \partial D_i \to \partial \tilde{D}_i$ to be an orientation preserving homeomorphism satisfying $f(x_{i,j}) = x_{i,j}^L$ for $j = 1,2$. Let f_i be an extension of $f_i|_{\partial D_i}$ to the entire disk. One can choose the maps $f_i|_{\partial D_i}$ consistently so that they glue together to a homeomorphism $f : S_{g,n+m} \to S([L])$. Since the maps f_i are unique up to homotopy, f is also unique up to homotopy.

We define $\Psi([H]) = [\varphi]$, where φ is a diffeomorphism approximating f. The map Ψ is well-defined and by construction it is inverse to Φ. □

We now extend this to $\mathfrak{Rad}^{\sim}$ by defining a fattening of $E\mathfrak{Rad}$ as follows:

Definition 4.25. The *fattening* $E\mathfrak{Rad}^{\sim}$ is defined as

$$E\mathfrak{Rad}^{\sim} := \{(([L],[H]),[\Gamma,\lambda,\tilde{H}]) \,|\, [\Gamma,\lambda] \in \mathcal{G}([L])\} \subset E\mathfrak{Rad} \times \mathcal{EMFat}^{ad}$$

where $\mathcal{G}([L])$ is the space given in Definition 4.7.

Recall that $E\mathfrak{Rad}$ consists of pairs $([L],[H])$ of a radial slit configuration and a marking, and that $\mathcal{EMFat}^{ad}$ consists of isomorphism classes of triples $[\Gamma,\lambda,H]$ of an admissible fat graph, a metric and a marking.

Corollary 4.26. *The projection* $E\mathfrak{Rad}^{\sim} \to \mathfrak{Rad}^{\sim}$ *is a universal* $\mathrm{Mod}(S_{g,n+m})$*-bundle over* $\mathfrak{Rad}^{\sim}$

Proof. This is clear since the diagram below is a pullback diagram and π_1 is a homotopy equivalence by 4.19.

$$\begin{tikzcd} E\mathfrak{Rad}^{\sim} \arrow[r, "\pi_1 \times \mathrm{id}"] \arrow[d] & E\mathfrak{Rad} \arrow[d] \\ \mathfrak{Rad}^{\sim} \arrow[r, "\simeq", "\pi_1"'] & \mathfrak{Rad} \end{tikzcd}$$

To see that it is a pullback, notice that the path from $[\Gamma, \lambda] \in \mathcal{G}([L])$ to the critical graph $[\Gamma_{[L]}]$ described in Lemma 4.8 determines a zigzag in $|\mathcal{F}at^{ad}|$ under the composite

$$\mathcal{G}([L]) \overset{\iota}{\hookrightarrow} \mathcal{MF}at^{ad} \xrightarrow{r(-,1)} |\mathcal{F}at^{ad}|$$

where ι is the inclusion and r is the map give on Lemma 3.12. Moreover, since $\mathcal{G}([L])$ is contractible, ι is an inclusion and $r(-,1)$ is a homotopy equivalence there is a contractible choice of zig-zags representing paths from $[\Gamma, \lambda]$ to $[\Gamma_{[L]}]$ in $\mathcal{G}([L])$. Therefore, by Remark 3.21, a marking of $[\Gamma_{[L]}]$, uniquely determines a marking of $[\Gamma]$ and vice versa. Thus, for $[\Gamma, \lambda] \in \mathcal{G}([L])$ giving a tuple $(([L],[H]),[\Gamma, \lambda, \tilde{H}]) \in E\mathfrak{Rad} \times \mathcal{EMF}at^{ad}$ is equivalent to giving either a triple $(([L],[H]),[\Gamma,\lambda])$ or a triple $([L],[\Gamma,\lambda,\tilde{H}])$. This shows that the diagram above is a pullback. □

We now describe a general result on universal bundles, which we use to conclude that π_2 is a homotopy equivalence.

Proposition 4.27. *Let $E \to B$ and $E' \to B'$ be universal principal G-bundles with B and B' paracompact spaces. Let $f : B \to B'$ be a continuous map. If $f^*(E')$ is isomorphic to E as a bundle over B, then f is a homotopy equivalence.*

Proof. For any space X we can build a diagram

$$\begin{tikzcd} {[X,B]} \arrow[r, "\cong"] \arrow[d, "f\circ -"'] & \{\text{principal } G\text{-bundles over } X\} \\ {[X,B']} \arrow[ur, "\cong"'] & \end{tikzcd}$$

This diagram commutes since $f^*(E') \cong E$. For $X = B'$ one gets that there is a $[g] \in [B', B]$ such that $[f \circ g] = [id_{B'}]$. Then, $g^*(E) \cong g^*(f^*(E')) = E'$, so we can repeat the argument and obtain that there is an $h \in [B, B']$ such that $[g \circ h] = [id_B]$. Finally, since $[h] = [f \circ g \circ h] = [f]$ then f and g are mutually inverse homotopy equivalences. □

Corollary 4.28. *The projection $\pi_2 : \mathfrak{Rad}^{\sim} \to \mathcal{MF}at^{ad}$ is a homotopy equivalence.*

Proof. This follows directly from the proposition by the same argument given in the proof of 4.26, since the following diagram is a pullback.

$$\begin{tikzcd} E\mathfrak{Rad}^{\sim} \arrow[r, "\pi_2 \times \mathrm{id}"] \arrow[d] & \mathcal{EMF}at^{ad} \arrow[d] \\ \mathfrak{Rad}^{\sim} \arrow[r, "\pi_2"'] & \mathcal{MF}at^{ad} \end{tikzcd}$$

□

5. Sullivan diagrams and the harmonic compactification

We now compare the harmonic compactification of radial slit configurations $\overline{\mathfrak{Rad}}$ and the space of Sullivan diagrams $\mathcal{SD}$, as defined in Definitions 2.16 and 3.16 respectively.

To do that, we remark that the $\overline{\mathfrak{URad}}$ is a subcomplex of $\overline{\mathfrak{Rad}}$. In particular, it is the subcomplex consisting of cells indexed by the subset $\Upsilon_{\mathfrak{U}}$ of Υ consisting of all combinatorial types of unilevel radial slit configurations. Remark that this make the projection $p : \overline{\mathfrak{Rad}} \to \overline{\mathfrak{URad}}$ cellular.

Proposition 5.1. *The space $\mathcal{SD}$ is homotopy equivalent to $\overline{\mathfrak{Rad}}$. In fact, there is a cellular homeomorphism between $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$.*

Proof. It is enough to show this for connected cobordisms. Recall that the harmonic compactification of the space of radial slit configurations $\overline{\mathfrak{Rad}}$ is homotopy equivalent to the space of unilevel radial slit configurations $\overline{\mathfrak{URad}}$ by Lemma 2.23, so it suffices to prove the second stronger statement.

Since in $\overline{\mathfrak{URad}}$ all annuli have the same outer and inner radius and all slits sit in the outer boundary, the annular chambers are superfluous information. Thus, the combinatorial type of a unilevel configuration is determined only by its radial chamber configuration. More precisely, two

univalent configurations $[L]$ and $[L']$ have the same combinatorial type if and only if they differ from each other only by the size of the radial chambers. Finally, the orientation of the complex plane and the positive real line, induce a total ordering of the radial chambers on each annulus.

Similarly, on a Sullivan diagram, the leaves of the boundary cycles and the fat structure at the vertices where they are attached give a total ordering of the edges on the admissible cycles. We say two Sullivan diagrams $[\Gamma]$ and $[\Gamma']$ have the same combinatorial data if they differ from each other only on the lengths of the edges on the admissible cycles. A *(non-metric) Sullivan diagram G* is an equivalence class of Sullivan diagrams under this relation. We will first show that a radial slit configuration and a Sullivan diagram are given by the same combinatorial data. That is, that there is a bijection

$$\begin{gathered}\Upsilon_{\mathfrak{U}} := \{\text{combinatorial types of unilevel radial slit configurations}\}\\ \updownarrow\\ \Lambda := \{\text{non-metric Sullivan diagrams}\}\end{gathered}$$

We define a map $f : \Upsilon_{\mathfrak{U}} \to \Lambda$ by $[\mathcal{L}] \mapsto G_{[\mathcal{L}],0}$ where $G_{[\mathcal{L}],0}$ is the underlying (non-metric) Sullivan diagram of a unfolded graph of $[\mathcal{L}]$. This map is well defined, since a slit or a parametrization point jumping along another slit corresponds to a slide of a vertex along an edge not belonging to the admissible cycle. For example the configurations in Figure 9 are mapped to the graphs in Figure 20.

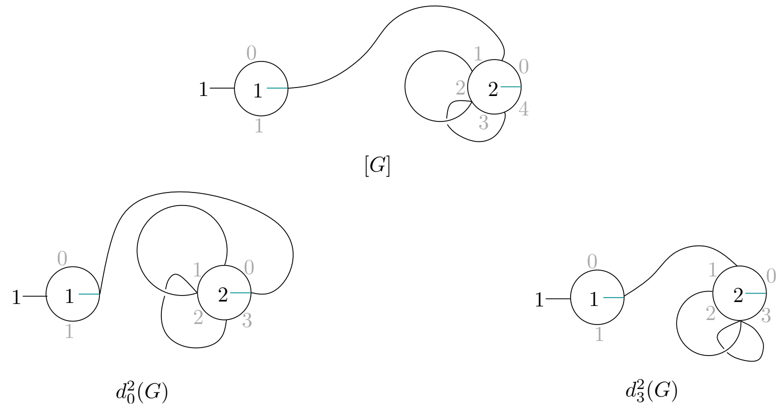

FIGURE 20. The top depicts a 5-cell which is a product of $\Delta^1 \times \Delta^4$-simplices in $\mathcal{SD}$, and the bottom two parts of its boundary. The edges are numbered in grey.

We next construct the inverse map $g : \Lambda \to \Upsilon_{\mathfrak{U}}$. Notice that any non-metric Sullivan diagram has a canonically associated metric Sullivan diagram by assigning all the edges in an admissible cycle the same length. Moreover any Sullivan diagram has a fat graph representative with all its vertices on the admissible cycles. A representative of a metric Sullivan diagram with all its vertices on the admissible cycles is given by the following data:

(i) A set of n parametrized circles $C_1, C_2, \ldots, C_n$ which are disjoint, ordered, and of length 1.
(ii) A finite number of chords $l_1, l_2, \ldots, l_s$ where a chord is a graph which consist of two vertices connected by an edge. Let V denote the set of vertices of such chords.
(iii) A subset $\widetilde{V} \subset V$ such that, $\widetilde{V}$ contains at least one vertex of each chord and $|V \backslash \widetilde{V}| = m$.
(iv) An assignment $\alpha : \widetilde{V} \to \sqcup_i C_i$ which will indicate how to attach the chords onto the n circles. Two or more chords may be attached on the same circle and even on the same point. The assignment α should attach at least one chord on each circle.
(v) For each x in the image of α, an ordering of the subset of chords attached to x, that is, an ordering of the set $\alpha^{-1}(x)$.

From this data one can construct a well defined metric fat graph with inner vertices of valence greater or equal to 3. The chords are attached onto the n circles using α. This gives the circles the structure of a graph by considering the attaching points as vertices and the intervals between them as edges. It just remains to give a fat structure at the attaching points. To do this let x be in the image of α. The parametrization of the circles gives a notion of incoming and outgoing half edges

on x say e_x^- and e_x^+ respectively. Moreover there is an ordering of the chords attached on x say $(l_{x,1}, l_{x,2}, \ldots, l_{x,s})$. The cyclic ordering at x is given by $(e_x^-, l_{x,1}, l_{x,2}, \ldots, l_{x,s}, e_x^+)$ as it is shown in Figure 21. Informally, this is to say all chords are attached on the outside of the circles according to the order given by the data. The chords that are attached only at one vertex give the leaves of the Sullivan diagram.

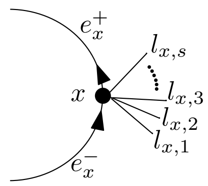

Figure 21. The fat structure induced at vertex x where the cyclic ordering is given by the orientation on the plane.

From this it is clear what the inverse map g should be. Given a Sullivan diagram G, its associated metric Sullivan diagram gives the data (i) to (v) listed above. Then, $g(G) = (\zeta, \lambda, \tilde{\omega}, \vec{r}, \vec{P})$ where ζ is given by α on the chords attached at both ends, λ is given by those chords (i.e. $\lambda(i) = k$ if and only if there is a chord attached on both ends connecting i and k), $\vec{P}$ is given by α on the chords attached only at one vertex, and $\tilde{\omega}$ and $\vec{r}$ are completely determined by the ordering of the chords at each attaching point. This map is well defined since slides along chords correspond to jumps along slits. Moreover, this map is clearly inverse to f.

We will show that $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$ have homeomorphic CW structures, where the cells are indexed by $\Upsilon_{\mathfrak{U}} \cong \Lambda$, by giving cellular homeomorphisms

$$\overline{\mathfrak{URad}} \xleftarrow{\varphi} \frac{\bigsqcup_{[\mathcal{L}]\in\Upsilon_{\mathfrak{U}}} e_{[\mathcal{L}]}}{\sim} \xrightarrow{\psi} \mathcal{SD}$$

We already saw the map φ in Definition 2.16. To construct the map ψ one must first notice that any Sullivan diagram $[\Gamma]$ in $\mathcal{SD}$ is completely and uniquely determined its non-metric underlying Sullivan diagram G and a tuple $(\vec{t}_1, \ldots, \vec{t}_{n_p})$ where t_{ij} is the length of the jth edge of the ith admissible cycle. Using this we can define

$$\psi(e_{[\mathcal{L}]}, (\vec{t}_1, \ldots, \vec{t}_{n_p})) = [\Gamma] = (f([\mathcal{L}]), (\vec{t}_1, \ldots, \vec{t}_{n_p}))$$

It is easy to show that the map ψ is continuous and by construction the homeomorphism $\varphi \circ \psi^{-1}$ is cellular with respect to the CW structures on $\overline{\mathfrak{URad}}$ and $\mathcal{SD}$. □

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